§4 Power Series

Goal use power series to get analytic functions.

Def. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence in $\mathbb{C}$.

Form the corresponding sequence of partial sums $\{S_N\}_{N=0}^{\infty}$ where

$$S_N := \sum_{n=0}^{N} a_n .$$

Then

(1) the series $\sum_{n=0}^{\infty} a_n$ converges to $a \in \mathbb{C} \Leftrightarrow \lim_{N\to\infty} S_N = a$,

(2) the series $\sum_{n=0}^{\infty} a_n$ diverges $\Leftrightarrow \{S_N\}_{N=0}^{\infty}$ diverges.

Remark As in $\mathbb{R}$

(1) $\sum_{n=0}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$.

(2) $\sum_{n=0}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=0}^{\infty} a_n$ converges.

Def. A power series in $(z-z_0)$ is a formal series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{with} \quad z_0 \in \mathbb{C} \text{ and } a_n \in \mathbb{C}.$$

Ex 4.1 $$\sum_{n=0}^{\infty} z^n = \begin{cases} \frac{1}{1-z} & |z|<1 \\ \text{dvg} & |z| \geq 1 \end{cases}$$

$$S_N = 1 + z + z^2 + \dots + z^N$$
$$z S_N = z + z^2 + \dots + z^N + z^{N+1}$$

$$(1-z) S_N = 1 - z^{N+1}$$

$$S_N \overset{z\neq 1}{=} \frac{1-z^{N+1}}{1-z} \qquad \text{etc} \dots$$

Thm 4.2 Weierstrass M-test

Let $G \subset \mathbb{C}$

$u_n : G \to \mathbb{C}$

$\sup_{z \in G} |u_n(z)| \le M_n$

$\sum_{n=0}^{\infty} M_n < \infty$

Then $\sum_{n=0}^{\infty} u_n(\cdot)$ converges uniformly on G.

Pf. Claim Fix $z \in G$. Then $\sum_{n=0}^{\infty} u_n(z)$ converges absolutely.

ETS $\left\{ \sum_{n=0}^{N} |u_n(z)| \right\}_{N=0}^{\infty}$ is Cauchy.

$$\left| \sum_{n=0}^{k} |u_n(z)| - \sum_{n=0}^{k+j} |u_n(z)| \right| = \sum_{n=k+1}^{k+j} |u_n(z)| \le \sum_{n=k+1}^{k+j} M_n \xrightarrow{k,j\to\infty} 0$$

So $\exists\ f : G \to \mathbb{C}$ s.t $f(z) := \sum_{n=0} u_n(z) \quad \forall z \in G$,

Fix $\varepsilon > 0$. Find $N \in \mathbb{N}$ s.t. $\sum_{k=N+1} M_k < \varepsilon$. Then $\forall n \ge N$

$$\sup_{z \in G} \left| f(z) - \sum_{k=0}^{n} u_k(z) \right| = \sup_{z \in G} \left| \sum_{k=n+1}^{\infty} u_k(z) \right|$$

$$\le \sup_{z \in G} \sum_{k=N+1}^{\infty} |u_k(z)| \le \sum_{k=N+1}^{\infty} M_k < \varepsilon. \quad \blacksquare$$

Def. Consider a power series

$$\sum_{n=0}^{\infty} c_n (z-a)^n . \qquad (*)$$

(1) The <u>Radius of Convergence</u> of $(*)$ is $R \in [0,\infty]$ given by

$$\frac{1}{R} := \overline{\lim_{n\to\infty}} |c_n|^{1/n} .$$

(2) The <u>circle of convergence</u> of $(*)$ is $\{z \in \mathbb{C} \mid |z-a| = R\}$.

<u>Thm 4.3</u> <u>Cauchy Root Test</u> Let

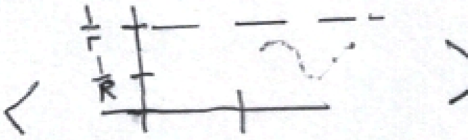
$$\sum_{n=0}^{\infty} c_n (z-a)^n \qquad (*)$$

be a power series with Radius of convergence R. Then

(1) $(*)$ converges absolutely for $|z-a| < R$.

(2) $(*)$ diverges for $|z-a| > R$.

(3) if $0 < r < R$, then $(*)$ converges uniformly on $\{z \in \mathbb{C} : |z-a| \le r\}$



for case $R \in (0,\infty)$

Pf. ^ LTGBG.

(1) Fix $z \in \mathbb{C}$ s.t. $|z-a| < r < R$.

Find $N \in \mathbb{N}$ st if $n \ge N$ then $|c_n|^{1/n} < \frac{1}{r}$. Then

$$|c_n (z-a)^n| = \left(|c_n|^{1/n} |z-a|\right)^n \le \left(\frac{|z-a|}{r}\right)^n = \alpha^n$$

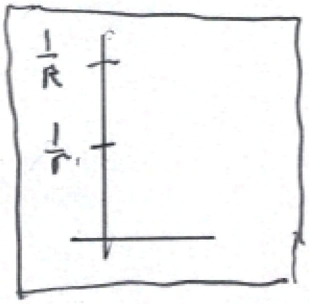
$$\boxed{0 \le \frac{|z-a|}{r} := \alpha < 1.}$$

So by Ex 4.1 (Geometric Series),

$(*)$ converges absolutely if $|z-a| < r < R$,

So (1) holds.

(2) Fix $z \in \mathbb{C}$ s.t. $|z-a| > r > R$.



Then $\exists$ infinitely many $n \in \mathbb{N}$ s.t.

$$|c_n|^{1/n} > \frac{1}{r}$$

$$\Downarrow$$

$$|c_n (z-a)^n| \geq \left(\frac{|z-a|}{r}\right)^n > 1$$

So $\lim\limits_{n\to\infty} c_n (z-a)^n \neq 0$.

So (*) diverges for $|z-a| > r$. So (2) holds

(3) Let $0 < r < s < R$. $\langle \frac{1}{R} < \frac{1}{s} < \frac{1}{r} \rangle$

Find $n \in \mathbb{N}$ s.t. if $n \geq N$ then $|c_n|^{\frac{1}{n}} < \frac{1}{s}$.

So $\forall\ n \geq N$ and $|z-a| < r$

$$|c_n (z-a)^n| = |c_n|\,|z-a|^n < \left(\frac{r}{s}\right)^n$$

But $0 < \frac{r}{s} < 1$ so (3) holds by Weierstrass-M-test. $\blacksquare$

<u>Thm 4.4</u> (Ratio Test). Consider a sequence $\{c_n\} \subset \mathbb{C} \setminus \{0\}$. Then

$$\varliminf_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| \leq \varliminf_{n\to\infty} |c_n|^{1/n} \leq \underbrace{\varlimsup_{n\to\infty} |c_n|^{1/n}}_{= \frac{1}{R} \text{ w/ } R \text{ the radius of convergence of } (*)} \leq \varlimsup_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$$

Thus, if $\lim\limits_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$ exists, then $\frac{1}{R} = \lim\limits_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$

<u>Pf.</u> See Introduction to Real Analysis by Manfred Stoll, p 290-291, Thm 7.1.10

Ex 4.5 Read @ home, (from Script)

Prop. 4.6 Let R be the radius of convergence of

$$\sum_{n=0}^{\infty} c_n (z-a)^n , \qquad (*)$$

Then R is also the radius of convergence of

$$\sum_{n=1}^{\infty} n c_n (z-a)^{n-1} \overset{\text{note}}{=} \sum_{n=0}^{\infty} (n+1) c_{n+1} (z-a)^n \qquad (**)$$

Pf. Recall $\lim_{n\to\infty} n^{1/n} = 1$. So

$$\overline{\lim_{n\to\infty}} |(n+1) c_{n+1}|^{1/n} = \left[\overline{\lim_{n\to\infty}} (n+1)^{\frac{1}{n+1}} |c_{n+1}|^{\frac{1}{n+1}} \right]^{\frac{n+1}{n}} =$$

$$= \left[1 \cdot \frac{1}{R} \right]^1 = \frac{1}{R}$$

Thm 4.7 Let $\sum_{n=0}^{\infty} c_n (z-a)^n$ have radius of convergence $R \neq 0$. Define

$$f : \underbrace{\{ z \in \mathbb{C} : |z-a| < R \}}_{:= B_R(a)} \to \mathbb{C}.$$

by

$$f(z) := \sum_{n=0}^{\infty} c_n (z-a)^n .$$

Then $f \in H(B_R(a))$ and

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z-a)^{n-1}$$

$\forall z \in B_R(a)$.

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Pf. Define $g : B_R(a) \to \mathbb{C}$ by

$$g(z) := \sum_{n=1}^{\infty} n c_n (z-a)^{n-1} .$$

<p.s.: can do by Prop 4.6. WTS $f'(z) = g(z) \ \forall z \in B_R(a)$.> WLOG, $a=0$.

Claim 1 $\quad (z+h)^n - z^n \overset{(1)}{=} h \sum_{k=1}^{n} (z+h)^{k-1} z^{n-k} \quad \forall z, h \in \mathbb{C}$. < usual convention of $0^0 = 1$

Idea of Pf $\quad h \sum_{k=1}^{n} (z+h)^{k-1} z^{n-k} = ((z+h)-z) \sum_{k=1}^{n} (z+h)^{k-1} z^{n-k}$

$$= \sum_{k=1}^{n} (z+h)^k z^{n-k} - \sum_{k=1}^{n} (z+h)^{k-1} z^{n-k+1}$$

cancellation heaven

$$= \sum_{k=1}^{n} (z+h)^k z^{n-k} - \sum_{k=0}^{n-1} (z+h)^k z^{n-k} = (z+h)^n - z^n .$$

Let $z, z+h \overset{(2)}{\in} B_r(0)$ with $0 < r < R$ and $h \neq 0$. Then

$$\left| \frac{f(z+h)-f(z)}{h} - g(z) \right| = \left| \sum_{n=1}^{\infty} c_n \underbrace{\left\{ \frac{(z+h)^n - z^n}{h} - n z^{n-1} \right\}}_{<=0 \text{ when } n=1>} \right| \quad \text{< by def of } f \,\&\, g \text{>}$$

$$\overset{(1)}{=} \left| \sum_{n=2}^{\infty} c_n \sum_{k=1}^{n} \underbrace{\left\{ (z+h)^{k-1} z^{n-k} - z^{n-1} \right\}}_{<=0 \text{ when } k=1>} \right| \quad \text{< used } \sum_{k=1}^{n} z^{n-1} = n z^{n-1} \text{>}$$

$$\overset{\Delta}{\leq} \sum_{n=2}^{\infty} |c_n| \sum_{k=2}^{n} \left| z^{n-k} \left((z+h)^{k-1} - z^{k-1} \right) \right|$$

$$\overset{(1)+\Delta}{\leq} |h| \sum_{n=2}^{\infty} |c_n| \left[\sum_{k=2}^{n} |z^{n-k}| \left(\sum_{l=1}^{k-1} |z+h|^{l-1} |z|^{k-1-l} \right) \right]$$

$$\overset{(2)}{\leq} |h| \sum_{n=2}^{\infty} |c_n| \left[\sum_{k=2}^{n} r^{n-k} \underbrace{\left(\sum_{l=1}^{k-1} r^{l-1} r^{k-1-l} \right)}_{=(k-1) r^{k-2}} \right]$$

$$= |h| \sum_{n=2}^{\infty} |c_n| \sum_{k=2}^{n} (k-1) r^{n-2}$$

$$= |h| \boxed{\sum_{n=2}^{\infty} |c_n| \tfrac{1}{2} n(n-1) r^{n-2}}$$

<- expression in box is finite by Prop. 4.6 (applied twice) since $r < R$

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<u>Cor 4.8</u> Let

$$f(z) := \sum_{n=0}^{\infty} c_n (z-a)^n$$

have radius of convergence $R \neq 0$. Then $\forall k \in \mathbb{N}$,

$$f^{(k)} : B_R(a) \to \mathbb{C}$$

exists so $f^{(k)} \in H(B_R(a))$ and furthermore, $\forall z \in B_R(a)$,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} c_n \, n(n-1)\cdots(n-k+1) \, (z-a)^{n-k} .$$

So

$$k! \, c_k = f^{(k)}(a) \quad , \text{ i.e. } \quad c_k = \frac{f^{(k)}(a)}{k!} .$$

Thus the coefficients c_k of a power series are unique.

<u>Ex 4.9</u> Consider the function $f : \mathbb{C} \to \mathbb{C}$

$$f(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \underset{=}{\text{i.e.}} \quad 1 + z^1 + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots .$$

<You should check the the ratio test yields that f has Rad. of Conv. $= \infty$.>

Thm 4.7 $\Rightarrow$

$$f'(z) = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = f(z) \qquad \forall z \in \mathbb{C}.$$

Define $h : \mathbb{C} \to \mathbb{C}$ by $h(z) = e^{-z} f(z)$. So

$$h'(z) = -e^{-z} f(z) + e^{-z} f(z) = 0 \qquad \forall z \in \mathbb{C}.$$

Prop 4.10 (to come next) $\Rightarrow$ h is constant on $\mathbb{C}$.

So $\forall z \in \mathbb{C}$

$$h(z) = h(0) = f(0) = 1,$$

i.e.

$$f(z) = e^z \qquad \forall z \in \mathbb{C}.$$

So

$$\boxed{e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \forall z \in \mathbb{C}}$$

<u>Example</u> Let G_1 and G_2 be open disjoint subsets of $\mathbb{C}$ and $G = G_1 \cup G_2$.

Define $f : G \to \mathbb{C}$ by

$$f(z) = \begin{cases} 1 & \text{if } z \in G_1 \\ 17 & \text{if } z \in G_2 . \end{cases}$$

Then $f \in H(G)$ and $f'(z) = 0 \quad \forall z \in G$.

But f is <u>not</u> constant on G.

<u>Prop 4.10</u> Let $G \subset \mathbb{C}$ be an open connected set.

Assume $f \in H(G)$ and $f'(z) = 0 \quad \forall z \in G$.

Then f is constant on G.

<u>Pf.</u> LTGBG. Fix $z_0 \in G$. Define

$$A := \{ z \in G \mid f(z) = f(z_0) \}. \qquad \langle \text{WTS } A = G \rangle$$

<u>Claim 1</u> $A \neq \emptyset$ b/c $z_0 \in A$.

<u>Claim 2</u> A is closed b/c f is cont. and $A = f^{-1}[\{f(x_0)\}]$.

<u>Claim 3</u> A is open

Fix $a \in A$. Then $\langle$b/c G is open$\rangle$ $\exists\, \varepsilon > 0$ st $B_\varepsilon(a) \subset G$. $\langle$ETS $B_\varepsilon(a) \subset A\rangle$

Fix $z \in B_\varepsilon(a)$. Define $g : [0,1] \to \mathbb{C}$ by

$$g(t) := f(tz + (1-t)a). \quad \langle \text{so } g \text{ is path from } f(a) \text{ to } f(z) \rangle$$

So for $t \in (0,1)$

$$g'(t) \overset{\text{chain rule = Prop 3.6}}{=} (z-a)\, f'(tz + (1-t)a) \overset{\text{assumption on } f'}{=} 0 .$$

So $\operatorname{Re} g, \operatorname{Im} g : [0,1] \to \mathbb{R}$ have deriv. 0. So g is constant on $[0,1]$

$$f(z) = g(1) = g(0) = f(a) \underset{a \in A := f^{-1}[\{f(z_0)\}]}{=} f(z_0),$$

and so $z \in A$. Thus $B_\varepsilon(a) \subset A$.

Thus, since G is <u>connected</u>, $A = G$. So f is constant on G. ∎