§ 1.3 Differentiable functions

Def 3.1 Let $G \subseteq \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$. Then f is (complex) differentiable at $z \in G$ provided

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h}$$

exists. When this limit exists, we denote it by $f'(z)$ and call it the (complex) derivative of f at z.

Also, f is called analytic (or holomorphic) on G if $f'(z)$ exists $\forall z \in G$.

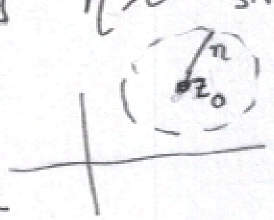
Notation $H(G) := \{ f : G \to \mathbb{C} \mid f \text{ is holomorphic on } G \}$.

here, G is an open subset of $\mathbb{C}$.

If S is any subset of $\mathbb{C}$, then f is holomorphic on S if $\exists$ open set G s.t. $G \supset S$ and $f \in H(G)$.

Rmk 3.2

1. Let $f : G \to \mathbb{C}$ be differentiable at $z_0 \in G$ w/ G open subset of $\mathbb{C}$. Then $\exists$ $\eta > 0$ s.t. (*) holds



$$N_\eta(z_0) := \{ z_0 + h \mid h \in \mathbb{C},\ |h| < \eta \} \overset{*}{\subset} G$$

$$\cup\!\!\text{I}$$

$$N'_\eta(z_0) := \{ z_0 + h \mid h \in \mathbb{C},\ 0 < |h| < \eta \}$$

So for $h \in \mathbb{C}$ and $0 < |h| < \eta$: $f(z_0 + h) = f(z_0) + h f'(z_0) + h\,\varepsilon(h)$

and $\lim \varepsilon(h) = 0$. So f is continuous at z_0.

2. Let $G \subseteq \mathbb{C}$ be open, $f: G \to \mathbb{C}$, $z_0 \in G$. Then

$$\left[f \text{ is differentiable at } z_0 \text{ w/ derivative } f'(z_0) \right] \iff$$

$$\left[\forall \varepsilon > 0 \ \exists \delta > 0 \text{ st if } h \in \mathbb{C} \text{ and } 0 < |h| < \delta \text{ then } \left| \frac{f(z_0+h) - f(z_0)}{h} - f'(z_0) \right| < \varepsilon . \right] \quad (*)$$

This motivates the next Theorem.

<u>Thm 3.3</u> <u>(Cauchy Riemann Equations).</u>

Let $G \subseteq \mathbb{C}$ be an open set and $f: G \to \mathbb{C}$ be differentiabe at $z = x+iy \in G$

Define $u, v : G \to \mathbb{R}$ by
$$u(x,y) = \operatorname{Re}[f(x+iy)]$$
$$v(x,y) = \operatorname{Im}[f(x+iy)]$$

$\langle$ so $f(x+iy) = u(x,y) + i v(x,y) \rangle$.

Then:

① the 1st order partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist at (x,y).

② These partials satisfy the Cauchy-Riemann Equations;

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

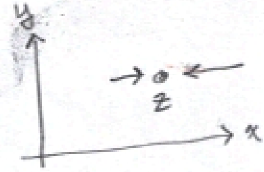
at the point (x,y).

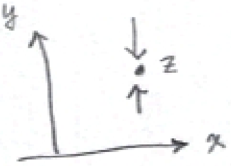
added → ③ $f'(x+iy) = u_x(x,y) + i\, v_x(x,y)$.

<u>Pf.</u> LTGBG.

<u>Idea</u> Have $z = x+iy$ & want to consider $z+h$ w/ $h \in \mathbb{C}$, $0<|h|<$ small.


a funky path $z+h$.





$\boxed{\text{For (a)}}$ path of form $(x+h)+iy$

$\boxed{\text{for (b)}}$ path of form $x+i(y+h)$.

Since f' exists at $z = x+iy$, we get by (*) applied:

(a) to $h \in \mathbb{R}$,

$$f'(z) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x+h,y)-u(x,y)}{h} + i\,\frac{v(x+h,y)-v(x,y)}{h}$$

$$= \left.\frac{\partial u}{\partial x}\right|_{(x,y)} + i\left.\frac{\partial v}{\partial x}\right|_{(x,y)}.$$

(b) ~~To~~ h of the form $h = ik$ w/ $k \in \mathbb{R}$

$$f'(z) = \lim_{\substack{k\to 0\\ k\in\mathbb{R}}} \frac{u(x,y+k)-u(x,y)}{ik} + i\,\frac{v(x,y+k)-v(x,y)}{ik}$$

$$= -i\left.\frac{\partial u}{\partial y}\right|_{(x,y)} + \left.\frac{\partial v}{\partial y}\right|_{(x,y)} = \left.\frac{\partial v}{\partial y}\right|_{(x,y)} + i\left.\left(\frac{-\partial u}{\partial y}\right)\right|_{(x,y)}$$

So the 1st order partials of u & v exist at (x,y).

Equating the 2 different expresion above for $f'(z)$ give the C-R Equations. ∎

<u>Reading Assigment</u> Ex 3.4 class notes, p 5.

<u>Def 3.5</u> A function $f:\mathbb{C}\to\mathbb{C}$ is entire $\iff$ f is analytic on $\mathbb{C}$

$\overset{\text{i.e.}}{\iff}$ $f \in H(\mathbb{C})$

<u>Prop 3.6</u> Let G, G_1 be nonempty open subsets of $\mathbb{C}$.

(1) If $f, g \in H(G)$ and $\lambda \in \mathbb{C}$, then $f+g$, λf, $fg \in H(G)$

$\quad\quad [(fg)(z) := f(z)\, g(z)]$.

(2) If $f \in H(G)$ and $g \in H(G_1)$ and $f(G) \subset G_1$,

<think: what $g \circ f : G \xrightarrow{f} G_1 \xrightarrow{g} \mathbb{C}$ >

then $g \circ f \in H(G)$ and $(g \circ f)'(z) = g'(f(z))\, f'(z)$ $\forall z \in G$.

Pf. Analogous to corresponding proofs for $\mathbb{R}$-valued case.

Use Remark 3.2. See class notes p. 6.

<u>Def / Cor 3.7.</u>

(1) A (complex) <u>polyonomial</u> $p : \mathbb{C} \to \mathbb{C}$ is a function of the form

$$p(z) = a_0 + a_1 z^1 + \dots + a_n z^n$$

for some $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\{a_j\}_{j=0}^{n} \in \mathbb{C}^{n+1}$.

Note $p \in H(\mathbb{C})$.

(2) A (complex) <u>rational function</u> f is of the form

$$f(z) = \frac{p(z)}{q(z)} \quad \text{where } p, q \text{ are polynomials.}$$

Note that since q is continuous, $\mathbb{C} \setminus (q^{-1}(\{0\})$ is open.

Note $f \in H(\mathbb{C} \setminus \{z \in \mathbb{C} \mid q(z) = 0\})$.

Remark mg 3.1

Let $z_0 = x_0 + iy_0 \in G \overset{\text{open}}{\subseteq} \mathbb{C}$.

Identify $G \subseteq \mathbb{C}$ with $G := \{(x,y) : x+iy \in G\} \subseteq \mathbb{R}^2$.

Note $G \subseteq \mathbb{C}$ is open $\iff$ $G \subseteq \mathbb{R}^2$ is open.

Consider $\underset{\mathbb{C}}{f : \underset{\cap}{G} \to \mathbb{C}}$ and $u, v : \underset{\mathbb{R}^2}{\underset{\cap}{G}} \to \mathbb{R}$ where

$$f(x+iy) = u(x,y) + i\, v(x,y).$$

The <u>Cauchy-Riemann Equations</u> for f are given by:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \qquad (\text{CR eq.})$$

Thm 3.3 says the following

If $[f$ is differentiable at $z_0] \Rightarrow [f$ satisfies the (CR eq.) at

<u>Ex mg 3.2</u> shows $\Leftarrow$ fails

<u>Ex mg 3.2</u> Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) := \begin{cases} \dfrac{(\bar{z})^2}{z} \overset{\text{i.e.}}{\equiv} \dfrac{x^3 - 3xy^2}{x^2+y^2} + i\,\dfrac{(y^3 - 3x^2 y)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

and the point $z_0 = 0$. Straightforward calculations show that

(1) $u_x(0,0) = 1 = v_y(0,0)$ and $u_y(0,0) = 0 = -v_x(0,0)$
$h+i0$

(2) $\displaystyle\lim_{\mathbb{R} \ni h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = 1.$ as $\to z_0 \leftarrow$ approach z_0 along x-axis

(3) $\displaystyle\lim_{\mathbb{R} \ni h \to 0} \frac{f(z_0 + (h+ih)) - f(z_0)}{h} = -1$ as $\nearrow z_0$ approach z_0 along line $x=y$.

So f satisfies the CR eq. at z_0 but is <u>not</u> differentiable at z_0.

Now, Thm 3.13 gives a "partial converse" of Thm 3.3.

Thm 3.13 ⟨Loosely speaking: [CR eq ⊕ cont 1st ∂] $\Rightarrow$ diff.⟩

Let $z_0 = x_0 + i y_0 \in G \overset{\text{open}}{\subseteq} \mathbb{C}$.

Let $f : G \to \mathbb{C}$ with $f = u + iv$ where $u = \operatorname{Re} f$ and $g = \operatorname{Im} f$
and $u, v : G \to \mathbb{R}$.

Let the partial derivatives u_x, u_y, v_x, v_y :

(1) satisfy the CR-eq. at z_0

(2) exists on G

(3) are continuous at z_0.

Then (4) f is differentiable at z_0

(5) $f'(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0)$.

$\Downarrow$(1)

(6) $f'(z_0) = v_y(x_0, y_0) - i\, u_y(x_0, y_0)$. ∎

Well, to prove Thm 3.13 we will use the following lemma.

Lemma 3.12

Let: $(x_0, y_0) \in G \overset{\text{open}}{\subset} \mathbb{R}^2$

$u : G \to \mathbb{R}$

$u_x, u_y : G \to \mathbb{R}$ exist and are continuous at (x_0, y_0)

Then $\exists\, N_\varepsilon((0,0)) \subset \mathbb{R}^2$ and function $\varepsilon_1, \varepsilon_2 : N_\varepsilon((0,0)) \to \mathbb{R}$ st,

for $h = (h_1, h_2) \in N_\varepsilon((0,0))$

(1) $u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0) = h_1 u_x(x_0, y_0) + h_2 u_y(x_0, y_0) + \underbrace{h_1 \varepsilon_1(h) + h_2 \varepsilon_2(h)}_{\text{think of as "error term"}}$

(2) $\lim\limits_{h \to (0,0)} \varepsilon_j(h) = 0$ for $j = 1, 2$.

Pf. This is a basic result from undergraduate Vector Calculus.
See Class Script, p 7, for a proof.

Proof of Thm 3.13 LTGBG. Identify $\mathbb{C}$ with $\mathbb{R}^2$.

Apply Lemma 3.12

(1) to u and find the corresponding $\varepsilon_1, \varepsilon_2 : N_{\varepsilon_u}((0,0)) \to \mathbb{R}$

(2) to v and find the corresponding $\varepsilon_3, \varepsilon_4 : N_{\varepsilon_v}((0,0)) \to \mathbb{R}$.

Let work on $N_\varepsilon((0,0))$ where $\varepsilon = \min(\varepsilon_u, \varepsilon_v)$. So, for $i = 1,2,3,4$

- $\varepsilon_j : N_\varepsilon((0,0)) \to \mathbb{R}$
- $\lim_{h \to 0} \varepsilon_j(h) = 0$.

Write $h = h_1 + i h_2 \in N_\varepsilon((0,0)) \underset{\text{so}}{\overset{\text{identified}}{\subseteq}} \mathbb{C}$. Let's compute:

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{f((x_0+h_1) + i(y_0+h_2)) - f(x_0+iy_0)}{h}$$

$$= \frac{[u(x_0+h_1, y_0+h_2) - u(x_0, y_0)] + i[v(x_0+h_1, y_0+h_2) - v(x_0, y_0)]}{h}$$

Lemma 3.12

$$= \frac{h_1}{h} u_x(x_0,y_0) + \frac{h_2}{h} \overbrace{u_y(x_0,y_0)}^{\boxed{-v_x(x_0,y_0) = i \cdot i v_x(x_0,y_0)}} + \frac{h_1}{h}\varepsilon_1(h) + \frac{h_2}{h}\varepsilon_2(h)$$

$$+ i\Big[\frac{h_1}{h} v_x(x_0,y_0) + \frac{h_2}{h} \underbrace{v_y(x_0,y_0)}_{\overset{\text{CR}}{=}\, u_x(x_0,y_0)} + \frac{h_1}{h}\varepsilon_3(h) + \frac{h_2}{h}\varepsilon_4(h)\Big]$$

$$= \underbrace{\frac{h_1+ih_2}{h}}_{=1} u_x(x_0,y_0) + i\Big[\underbrace{\frac{h_1+ih_2}{h}}_{=1} v_x(x_0,y_0)\Big] + \underbrace{\frac{h_1}{h}}_{\left|\frac{h_1}{h}\right| \le 1}\varepsilon_1(h) + \frac{h_2}{h}\varepsilon_2(h) + \frac{ih_1}{h}\varepsilon_3(h) + \underbrace{\frac{ih_2}{h}}_{\left|\frac{h_2}{h}\right| \le 1}\varepsilon_4(h)$$

$$\xrightarrow{h \to 0} u_x(x_0,y_0) + i v_x(x_0,y_0).$$

∎

Here is the "full story"

Thm 3.11

Let $z_0 = x_0 + iy_0 \in G \overset{\text{open}}{\subset} \mathbb{C}$.

Let $f: G \to \mathbb{C}$ with $f = u + iv$ where $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$.

Then TFAE.

(i) f is (complex) differentiable at z_0

(ii) f is real differentiable at z_0 and the CR-eq. hold at (x_0, y_0)

↳ See Class Script Def 3.8.

Remark You are not responsible for Thm 3.11. I just thought you might want to see an TFAE statement between (complex) differentibilty an CR-eq. You can read up on this in Class Script p 6-7.

Remark The below corollary to Thm 3.13 show "just how right/clever etc" the definition of $e^{x+iy} := e^x \cos y + i e^x \sin y$ is.

Cor 3.14 Let $f: \mathbb{C} \to \mathbb{C}$ be given by $f(z) = e^z$. Then:

(1) f is entire (i.e $f \in H(\mathbb{C})$)

(2) $f'(z) = e^z \quad \forall z \in \mathbb{C}$.

(Sketch of) Proof. Note $f = u + iv$ where $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$.

By Thm 3.13, it suffices to show that f satisfies the CR-eq. on all of $\mathbb{C}$.

Claim $u_x = v_y$.

A direct calculation gives $u_x = e^x \cos y$ and $v_y = e^x \cos y$. ☺

Claim $u_y = -v_x$

A direct calculation gives $u_y = -e^x \sin y$ and $v_x = e^x \sin y$ ☺

Claim. u_x, v_y, u_y, v_x are continuous on $\mathbb{C}$.

Prop 3.15

Let: $G_1, G_2 \overset{\text{open}}{\subset} \mathbb{C}$ $\qquad \left\langle \text{So} \quad G_1 \underset{g}{\overset{f}{\rightleftarrows}} G_2 \right\rangle$

$f: G_1 \to G_2$ be continuous

$g: G_2 \to G_1$ be continuous, $g \in H(G_2)$, $g'(z) \neq 0 \ \forall z \in G_2$.

$g(f(z)) = z \ \forall z \in G_1$

Then (1) $f \in H(G_1)$

(2) $f'(z_0) = \dfrac{1}{g'(f(z_0))} = \dfrac{1}{(g' \circ f)(z_0)} \quad \forall z_0 \in G_1$

Pf. LTGB G.

Claim f is 1-to-1.

Fix $z_1, z_2 \in G_1$ s.t. $f(z_1) = f(z_2)$. $\langle$ WTS $z_1 = z_2$. $\rangle$

$\Downarrow$

$z_1 \overset{\text{given}}{=} (g \circ f)(z_1) = (g \circ f)(z_2) \overset{\text{given}}{=} z_2$

Fix $z_0 \in G_1$. Find $\varepsilon > 0$ st $N_\varepsilon(z_0) \subset G_1$. Then $\langle z \in N_\varepsilon'(z_0) \xrightarrow[\text{1-to-1}]{f \text{ is}} f(z) - f(z_0) \neq$

$$\underset{\forall z \in N_\varepsilon'(z_0)}{1 \uparrow=} \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} \cdot \frac{f(z) - f(z_0)}{z - z_0} \xrightarrow{\text{lim as } z \to z_0} g'(f(z_0)) \cdot f'(z_0)$$

Cor 3.16 Let $G \subseteq \mathbb{C}$ be open and connected

$f: G \to \mathbb{C}$ be a branch of the logarithm on G.

Then (1) $f \in H(G)$

(2) $f'(z) = \dfrac{1}{z} \quad \forall z \in G$.

Pf. Apply Prop 3.15 w/ $g(z) = e^z$.

Def. Let $G \overset{\text{open}}{\subset} \mathbb{R}^2$

$u: G \to \mathbb{R}$.

Then u is <u>harmonic</u> on G provided:

(1) 1^{st} & 2^{nd} order partial derivatives of u exist and are continuous on G

(2) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ on G

Laplace's equation.

Rmk ① Let $f = u + iv \in H(G)$ w/ $G \overset{\text{open}}{\subseteq} \mathbb{C}$. < so CReq hold on G (Thm 3.

② Let u and v have continuous 2^{nd} order partial derv.

< late we will show $f \in H(G) \Rightarrow$ this holds >

well then

③ $\dfrac{\partial u}{\partial x} \overset{CR}{=} \dfrac{\partial v}{\partial y} \Longrightarrow$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \overset{CR}{=} \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \Rightarrow u \text{ is harmonic.}$$

④ Similarly v is harmonic on G.

Def. Let $G \overset{\text{open}}{\subseteq} \mathbb{R}^2$. Functions $u, v : G \to \mathbb{R}$ are <u>harmonic conjugates</u> on G provided.

(1) u and v are harmonic on G,

(2) $u + iv \in H(G)$.

<u>So</u> For $u, v : G \to \mathbb{R}$ w/ $G \overset{\text{open}}{\subseteq} \mathbb{R}^2$:

u is the harmonic conjugate of v

$\iff u + iv \in H(G)$.

<u>Thm</u> Let $N_\varepsilon((x_0, y_0)) := D \overset{\text{open}}{\subseteq} \mathbb{R}^2$.

Let $u : D \to \mathbb{R}$ be haromic.

Then $\exists\ v : D \to \mathbb{R}$ st v is harmonic & $u + iv \in H(D)$.

<u>Sketch of Proof</u> <Vector Calculus>

Consider the differential
$$\underset{\displaystyle -\frac{\partial u}{\partial y}}{\underset{\shortparallel}{P}}\, dx + \underset{\displaystyle \frac{\partial u}{\partial x}}{\underset{\shortparallel}{Q}}\, dy$$

Since u is harmonic;

(1) P and Q have cont. partial derv. in D

(2) $$\underset{\displaystyle -\frac{\partial^2 u}{\partial y^2}}{\underset{\shortparallel}{\frac{\partial P}{\partial y}}} = \underset{\displaystyle \frac{\partial^2 u}{\partial x^2}}{\underset{\shortparallel}{\frac{\partial Q}{\partial x}}}$$

So By advanced calculus, $\exists\ v : D \to \mathbb{R}$ st

$$dv = P\,dx + Q\,dy.$$

So $$\frac{\partial v}{\partial x} = P \overset{\text{def}}{=} -\frac{\partial u}{\partial y} \quad \& \quad \frac{\partial v}{\partial y} = Q \overset{\text{def}}{=} \frac{\partial u}{\partial x}.$$

So $u + iv \in H(D)$ <by Thm 3.13: [CR ⊕ cont 1st ∂] ⇒ diff>.

In fact $$v(\tilde{x}, \tilde{y}) = \int_{(x_0, y_0)}^{(\tilde{x}, \tilde{y})} [-u_y(x, y)\,dx + u_x(x, y)\,dy].$$

A line integral.

Example Construct an analytic function whose real part is

$$u(x,y) = x^3 - 3xy^2 + y.$$

Claim u is harmonic on $\mathbb{C}$.

$u_x = 3x^2 - 3y^2$ | $u_y = -6xy + 1$

$u_{xx} = 6x$ | $u_{yy} = -6x$

$\frac{\partial}{\partial y} u_x = -6y$ | $\frac{\partial}{\partial x} u_y = -6y$

So 1^{st} & 2^{nd} partial derv. are cont. on $\mathbb{C}$.

Let's check Laplace's eq.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

Now we want to find the harmonic conjugate v of u.

(1) $\frac{\partial v}{\partial y} \stackrel{CR}{=} \frac{\partial u}{\partial x} = 3x^2 - 3y^2$

(2) $\frac{\partial v}{\partial x} \stackrel{CR}{=} -\frac{\partial u}{\partial y} = 6xy - 1$.

(3) $v(x,y) = \int \frac{\partial v}{\partial y}\, dy \stackrel{(1)}{=} \int (3x^2 - 3y^2)\,dy = 3x^2y - y^3 + g(x)$

find him next

(4) $6xy - 1 \stackrel{(2)}{=} \frac{\partial v}{\partial x} \stackrel{(3)}{=} 6xy + g'(x) \Rightarrow g'(x) = -1$

(5) $g'(x) = -1 \Rightarrow g(x) = -x + c$.

So $v(x,y) = 3x^2y - y^3 - x + c$.