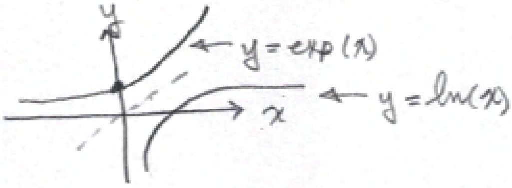
§ 1.2 Elementary transcental functions.

Remark 1.2.1 Consider the function $\exp(\cdot) : \mathbb{R} \to \mathbb{R}$.



The function $\exp(\cdot) : \mathbb{R} \to (0,\infty)$ is 1 to 1 and (now) onto so we can define it's inverse $\ln(\cdot) : (0,\infty) \to \mathbb{R}$.

Trouble $\exp(\cdot) : \mathbb{C} \to \mathbb{C}$ as we defined as $\exp(x+iy) = e^x(\cos y + i \sin y$ is not 1-to-1, for indeed, $\forall k \in \mathbb{Z}$,

$$\exp(x+iy) = e^x(\cos(y+2\pi k) + i\sin(y+2\pi k)) = \exp(x+i(y+2\pi$$

But we want to define an inverse of exp. fn. so we need to restrict the domain of exp. fn so can get an inverse. So here we go

Def. 1.2.2 Consider $z = x+iy = re^{i\theta} \in \mathbb{C}/\{0\}$ and $\boxed{r>0}$. <ps: $z=0$ is trouble-maker.>

Then the argument of z = $\arg z$ is the set

$$\arg z := \{\theta + 2\pi k : k \in \mathbb{Z}\}$$

The principal value of argument of z = $\operatorname{Arg} z$ is the real number uniquely defined by

$$\operatorname{Arg} z \in \arg z \quad \text{and} \quad -\pi < \operatorname{Arg} z \leq \pi .$$

So $\boxed{\text{if } \theta \in \arg z, \text{ then } z = |z| e^{i\theta}.}$

So, going back to the Example in 1.1.6, were, for $k \in \mathbb{Z}$

$$z = -\sqrt{2} + i\sqrt{2} = 2e^{i(\frac{3\pi}{4} + 2\pi k)} = -2e^{i(-\pi/4 + 2\pi k)},$$

$$\arg(-\sqrt{2} + i\sqrt{2}) = \{\tfrac{3\pi}{4} + 2\pi k \mid k \in \mathbb{Z}\}$$

and

$$\operatorname{Arg}(-\sqrt{2} + i\sqrt{2}) = \frac{3\pi}{4}.$$

Note from Compute 1.1.5 (1), $\arg(z_1 z_2) = \arg z_1 + \arg z_2$, as sets

Compute Fix $w \in \mathbb{C}$.

going for $z = \log w$ but will get into trouble.

Find $\{ z \in \mathbb{C} \mid e^z = w \}$ $\underline{\underline{\text{ie.}}}$ $\{ x+iy \mid e^{x+iy} = w \}$.

Well write $z = x+iy$ and compute:

$w = e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$

$|w| = |e^x| \, |\cos y + i \sin y| = e^x > 0 \quad \Rightarrow \quad |w| \neq 0 \Rightarrow w \neq 0$

$\Downarrow$

$x = \log |w|$

y is any element in the set $\{ \operatorname{Arg} w + 2\pi k \mid k \in \mathbb{Z} \}$.

So

$\{ z \in \mathbb{C} \mid w = e^z \} = \{ \log |w| + i(\operatorname{Arg} w + 2\pi k) \mid k \in \mathbb{Z} \}$

"think"

Def 2.1 Let $G \subset \mathbb{C}$ be an open connected set and $f : G \to \mathbb{C}$ be a continuous function such that

$z = e^{f(z)} \quad \forall z \in G.$

Then f is a branch of the logarithm on G.

Note Let f be a branch of the logarithm on G. Then:

(1) $0 \notin G$

(2) $f(z) = \log |z| + i(\operatorname{Arg} z + 2\pi k_z) \quad \exists k_z \in \mathbb{Z}$.

(3) $g(z) := f(z) + i2\pi k$ is a branch of the log on G $\forall k \in \mathbb{Z}$.

Prop 2.2 Let $G \subseteq \mathbb{C}$ be an open connected set.

Let $g, f : G \to \mathbb{C}$ be branches of the log on G.

Then $\exists\, k \in \mathbb{Z}$ st $\forall\, z \in G$; $g(z) = f(z) + i 2\pi k$.

pf. LTGBG.

Define $h : G \to \mathbb{C}$ by $h(z) = \dfrac{f(z) - g(z)}{2\pi i}$

So $\underbrace{h \text{ is continuous on } G}$ and $\underbrace{h(G) \subseteq \mathbb{Z} \subseteq \mathbb{C}}$

$\Downarrow$ G is connected $\qquad\qquad \downarrow$

$h(G)$ is connected$] \longrightarrow h(G) = k \;\; \exists\, k \in \mathbb{Z}$.

Remark. Let f be a branch of the log on a open connected $G \subset \mathbb{C}$.

So $f(z) = \log|z| + i\,(\operatorname{Arg} z + 2\pi k_z) \quad \exists\, k_z \in \mathbb{Z}$.

Rewrite $f(z) = \log|z| + i\,\theta(z)$ w/ $\theta : G \to \mathbb{R}$.

$\uparrow$

he is a selection of arg z from $\{\operatorname{Arg} z + 2\pi k \mid k \in \mathbb{Z}\}$

$\left[\begin{array}{l} f \text{ is cont. by def} \\ \log|z| \text{ is cont.} \end{array}\right] \Rightarrow \theta : G \to \mathbb{R}$ must be continuous

$\Downarrow$ G is connected

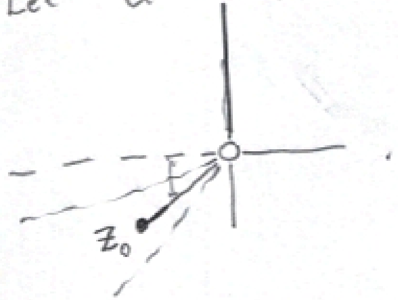
$\theta(G)$ is an interval.

Also, length $\theta(G) \leq 2\pi$.

So to find a branch of the log on G, one needs to find such a cont. selection θ on G.

Ex 2.3

(i) Let $G = \mathbb{C} \setminus \{z \in \mathbb{R} : z \le 0\}$. Note G is open & connected.



Claim $f : G \to \mathbb{C}$ via

$f(z) = \log|z| + i \operatorname{Arg} z$ is a branch of the $\log z$ on G.

Pf. Enough to show $\operatorname{Arg} : G \to \mathbb{C}$ is continuous.

Let $G \ni z_n \xrightarrow{n \to \infty} z_0 \in G$.

Fix $\varepsilon > 0$. Know $-\pi < \operatorname{Arg} z_0 < \pi$.

Find $\delta > 0$ s.t $\delta < \varepsilon$ and the sector

$$\Sigma_\delta(z_0) := \{z \in G : |\operatorname{Arg} z - \operatorname{Arg} z_0| < \delta\} \subseteq G.$$

Find N st if $n \ge N$ then $z_n \in \Sigma_\delta(z_0)$.

So if $n \ge N$ then $|\operatorname{Arg} z_n - \operatorname{Arg} z_0| < \varepsilon$.

Def. This branch is called the principal branch of $\log z$ and is denoted by $\operatorname{Log} z$. So

$$\operatorname{Log}(\cdot) : \mathbb{C} \setminus \{z \in \mathbb{R} : z \le 0\} \to \mathbb{C}$$

$$\boxed{\operatorname{Log}(z) = \log|z| + i \operatorname{Arg} z}$$

(ii) Let $G = \mathbb{C} \setminus \{z \in \mathbb{R} : z \ge 0\}$.

Finde $\theta : G \xrightarrow{\text{conti}} \mathbb{C}$ s.t. $\theta(z) \in \arg(z) \cap (0, 2\pi)$

Then $f(z) = \log|z| + i\theta(z)$ is a branch of the $\log z$ on G.