# Chapter 1 Holomorphic (ie. Analytic Functions).

## §1.1 Definitions and elementary properties

**Def 1.1.1** The complex plane $\mathbb{C}$ can be defined as the set
**Viewpoint**

$$\mathbb{C} = \{(x,y) : x, y \in \mathbb{R}\},$$

endowed with addition & multiplication $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \qquad (a)$$

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1). \qquad (m)$$

Well, this viewpoint is mathematically sound but not "practical."

Identify (let's temporarily use the notation $\hat{=}$) $(0,1) \in \mathbb{C}$ by $i$, ie $\boxed{i \hat{=} (0,1)}$

So $i^2 \hat{=} (0,1)(0,1) \overset{(m)}{=} (0-1, 0-0) = (-1, 0)$.

Likewise, for $x, y \in \mathbb{R}$,

$$\mathbb{R} \ni x \hat{=} (x, 0) \in \mathbb{C}$$

$$\mathbb{R} \ni y \hat{=} (y, 0) \in \mathbb{C}.$$

So, using (a) and (m), we get

$$\begin{aligned} x + iy &= (x,0) + (0,1)(y,0) \\ &\overset{(m)}{=} (x,0) + (0 \cdot y - 1 \cdot 0, 0 + y) = (x,0) + (0,y) \\ &\overset{(a)}{=} (x,y). \end{aligned}$$

So henceforth we will view $\mathbb{C}$ as

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$$

where $i^2 = -1$. Note (a) & (m) are now "natural", ie

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \qquad (a)$$

$$(x_1 + iy_1)(x_2 + iy_2) \overset{\text{"foil"}}{=} (x_1 x_2 + \underbrace{i^2}_{-1} y_1 y_2) + i(x_1 y_2 + x_2 y_1) \qquad (m)$$

Def. 1.1.2 Let $z = x+iy \in \mathbb{C}$. < Henceforth, it's understood that $x, y \in \mathbb{R}$.> Then:

(1) Real part of $z$ = $\operatorname{Re} z := x$. & note $\operatorname{Re} z \in \mathbb{R}$.

(2) Imaginary part of $z$ = $\operatorname{Im} z := y$. & note $\operatorname{Im} z \in \mathbb{R}$.

(3) modulus of $z$ or absolute value of $z$ = $|z| := \sqrt{x^2+y^2}$

(4) The distance between $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$ is

$$d(z_1,z_2) = d((x_1+iy_1),(x_2+iy_2)) = |(x_1+iy_1) - (x_2+iy_2)|$$

$$\overset{(a)}{=} |(x_1-x_2)+i(y_1-y_2)| \overset{(3)}{=} \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.$$

Rmk. 1.1.3 (1) We endow $\mathbb{C}$ with the metric $d(z_1,z_2)$ as given above.

(2) If $S \subseteq \mathbb{C}$, with endow $S$ w/ the usual subspace metric.

So we can consider:

- sequences $\{z_n\}$ from $\mathbb{C}$ and talk about limits
- functions $f: S \to \mathbb{C}$ and talk about continuity.
- etc...

Def 1.1.4 Define a function $\exp: \mathbb{C} \to \mathbb{C}$
<1.5.1>

$$\exp(x+iy) \overset{\text{ie}}{=} e^{x+iy} := e^x \cos y + i\, e^x \sin y$$

$$\overset{\text{ie.}}{=} e^x(\cos y + i \sin y).$$

Note $\exp|_{\mathbb{R}} : \mathbb{R} \to \mathbb{C}$ is the usual exp. fn. on $\mathbb{R}$ since

$$\exp(x+i0) = e^x(\cos 0 + i \sin 0) = e^x \in \mathbb{R}.$$

Note $e^{i\theta} \overset{\theta \in \mathbb{R}}{=} e^{0+i\theta} = e^0(\cos\theta + i\sin\theta) = \cos\theta + i\sin\theta$

so $|e^{i\theta}| \overset{\theta \in \mathbb{R}}{=} \sqrt{\cos^2\theta + \sin^2\theta} = 1.$

Thus $|re^{i\theta}| = |r|\,|e^{i\theta}| = |r|.$

$r, \theta \in \mathbb{R}$ and using $|z_1 z_2| = |z_1||z_2|$

So $re^{i\theta} \overset{(*)}{=} r\cos\theta + ir\sin\theta.$

Compute 1.1.5: As always, $r, r_i, \theta, \theta_i, x, y \in \mathbb{R}$

(1) $[r_1 e^{i\theta_1}][r_2 e^{i\theta_2}] = [r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)]$

$= [r_1 r_2][\cos\theta_1 + i\sin\theta_1][\cos\theta_2 + i\sin\theta_2]$

$= [r_1 r_2][\{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\} + i\{\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\}]$

$\overset{\text{trig}}{\underset{\text{ID}}{=}} [r_1 r_2][\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$= r_1 r_2 e^{i(\theta_1+\theta_2)}$ ← cool, what it "should be"

(2) $e^{x+iy} \overset{\text{def}}{=} e^x(\cos y + i\sin y) = e^x(e^0\cos y + i\,e^0\sin y)$

$\overset{\text{def}}{=} e^x(e^{0+iy}) = e^x e^{iy}$ ← as it "should be"

(3) $|e^{x+iy}| = |e^x(\cos y + i\sin y)| = |e^x|\,|\cos y + i\sin y| = e^x\sqrt{\cos^2 y + \sin^2 y} = e$

Picture Time 1.1.6

You can identify (for help w/ intuition)

$\mathbb{C} = \{x+iy \mid x, y \in \mathbb{R}\} \overset{\text{or}}{=} \{re^{i\theta} : r, \theta \in \mathbb{R}\}$

with

$\mathbb{R}^2 = \{\underbrace{(x,y)}_{\text{Cartesian Coords}} : x, y \in \mathbb{R}\} \overset{\text{or}}{=} \{\underbrace{(r,\theta)}_{\text{polar coords.}} : r, \theta \in \mathbb{R}\}$

where

$\mathbb{C} \ni x+iy \;\hat{=}\; (x,y) \in \mathbb{R}^2$ ← Cartesian coords, representation unique

$\mathbb{C} \ni re^{i\theta} \;\hat{=}\; (r,\theta) \in \mathbb{R}^2$ ← polar coords, representation not unique.

$= r\cos\theta + ir\sin\theta.$

recall: can take, well, if $re^{i\theta} \neq 0$:

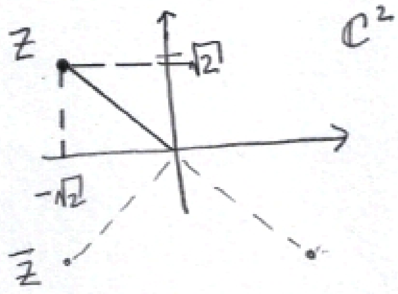
$r = |z| = |r| > 0$

$\tan\theta \underset{k\in\mathbb{Z}}{\overset{\text{note}}{=}} \tan(\theta + 2\pi k) = \frac{y}{x}$

need to take care — so not unique

note

So for $z = -\sqrt{2} + i\sqrt{2}$




$\mathbb{C}^2$

- $|z| = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2} = 2$
- $\tan\theta = \dfrac{\sqrt{2}}{-\sqrt{2}} = -1$
- $r>0$ & see want $\frac{\pi}{2} \le \theta \le \pi$ so $\theta = \frac{3\pi}{4}$

so $z = -\sqrt{2} + i\sqrt{2} = 2e^{i\,3\pi/4} \underset{k\in\mathbb{Z}}{\overset{\text{or}}{=}} 2e^{i\left(\frac{3\pi}{4}+2\pi k\right)}$

$\overset{\text{or}}{=} -2e^{i(-\pi/4)} \underset{k\in\mathbb{Z}}{\overset{\text{or}}{=}} -2e^{i(-\pi/4+2\pi k)}$.

Well, Let's think about the conjugate $\bar{z}$ of $z = x+iy$ $= re^{i\theta}$

Def $\bar{z} \overset{\text{ie}}{=} \overline{(x+iy)} := x - iy$

So $\bar{z} \overset{\text{ie}}{=} \overline{re^{i\theta}} = r\cos\theta - ir\sin\theta = r\cos(-\theta) + ir\sin(-\theta) = re^{-i\theta}$.

**Remark 1.1.7** References

(1) Our Book

(2) Complex Variables by Ash & Novinger

(3) Complex Variables Schaum's Outline.

**Note 1.1.8** Misc. Important Remarks

(1) $\mathbb{C} \ni x_n + iy_n \xrightarrow{n\to\infty} x+iy \in \mathbb{C} \iff \left[\begin{array}{l} \mathbb{R} \ni x_n \xrightarrow{n\to\infty} x \in \mathbb{R} \\ \mathbb{R} \ni y_n \xrightarrow{n\to\infty} y \in \mathbb{R} \end{array}\right]$

Indeed,

$$d(x_n+iy_n,\, x+iy) = |(x_n+iy_n) - (x+iy)| = \sqrt{(x_n-x)^2 + (y_n-y)^2}$$

(2) Consider a function $f : S \to \mathbb{C}$ where $S \subset \mathbb{C}$.

Define $\hat{S} = \{(x,y) \in \mathbb{R}^2 \mid x+iy \in S\}$ and

$u : \hat{S} \to \mathbb{R}$ by $u(x,y) = \mathrm{Re}[f(x+iy)]$

$v : \hat{S} \to \mathbb{R}$ by $v(x,y) = \mathrm{Im}[f(x+iy)]$.

Thus $f(x+iy) = u(x,y) + iv(x,y)$. ← Common Notation.

Note $[f \text{ is continuous}] \iff [u \text{ and } v \text{ are continuous}]$

Recall, a function is cont. iff it preserves convergent sequences

Using idea from part (1), for $S \ni x_n+iy_n \xrightarrow{n\to\infty} x+iy \in S$,

$|f(x_n+iy_n) - f(x+iy)| = \sqrt{(u(x_n,y_n) - u(x,y))^2 + (v(x_n,y_n) - v(x,y))^2}$

etc...

(3) Example of continuos $f : \mathbb{C} \to \mathbb{C}$.

- $f(z) = |z|$. $f(x+iy) = |x+iy| = \sqrt{x^2+y^2} = u(x,y)$
- $f(z) = e^z$. $f(x+iy) = \underbrace{e^x \cos y}_{u(x,y)} + i\underbrace{e^x \sin y}_{v(x,y)}$.

(4) $[\mathbb{C}\setminus\{0\} \ni r_n e^{i\theta_n} \xrightarrow{n\to\infty} re^{i\theta} \in \mathbb{C}\setminus\{0\}] \Rightarrow [|r_n| \to |r|]$

LTGBG. So $|r_n| = |r_n e^{i\theta_n}| \xrightarrow[\text{by (3)}]{n\to\infty} |re^{i\theta}| = |r|$.

(5) $[\mathbb{C}\setminus\{0\} \ni r_n e^{i\theta_n} \xrightarrow{n\to\infty} re^{i\theta} \in \mathbb{C}\setminus\{0\}] \not\Rightarrow [\theta_n \to \theta]$

Indeed, let $r_n = r = 17$, $\theta_n = 0$, $\theta = 2\pi$.