Homework 27

Define $f\colon \mathbb{C} \to \mathbb{C}$ and $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned} f(z) &:= \sqrt{|xy|} \qquad \text{where } x := \operatorname{Re} z \quad , \quad y := \operatorname{Im} z \\ u(x,y) &= \operatorname{Re} f(x+iy) \\ v(x,y) &= \operatorname{Im} f(x+iy) \ . \end{aligned}$$

Show that

1. u and v satisfies the Cauchy Riemann equations at $(x, y) = (0, 0)$
2. f is not differentiable at $z = 0$.

Proof. Your proof goes here. □

Solution. Your solution goes here. □

Homework 28

Let G be the unit disk in $\mathbb{C}$, i.e.

$$G = B_1(0) = \{z \in \mathbb{C}\colon |z| < 1\} .$$

Let $f \in H(G)$.

1. Show that if $\operatorname{Re} f$ is constant on G then f is constant of G.
2. Show that if e^f is constant on G then f is constant of G.

Homework 29

Let G be an open subset of $\mathbb{C}$ and $f \in H(G)$. Define

$$\begin{aligned} G^* &:= \{z \in \mathbb{C}\colon \overline{z} \in G\} \\ f^*(z) &:= \overline{f(\overline{z})} \qquad \text{for } z \in G^* \ . \end{aligned}$$

Note (i.e., you need not show) that G^* is open in $\mathbb{C}$.

1. Show that $f^* \in H(G^*)$.
2. Express $(f^*)'$ in terms of f'.

Homework 30

Problem Source: Quals 2004.
Let G be an open connected subset of $\mathbb{C}$.
Let $f \in H(G)$ be s.t. there is a constant $K \in \mathbb{R}$ with $|f(z)| = K$ for each $z \in G$.
Show that f is constant of G.

Homework 31

Let $\sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R.
What is the radius of convergence of the following two power series?

1. $\sum_{n=0}^{\infty} a_n \, (2z)^n$
2. $\sum_{n=0}^{\infty} (a_n)^2 \, z^n$

You may use, without proving, Lemma 0.1 provided you read the proof I provided below. But first, read the handout *lim sup and lim inf of sequences*, which is posted on our Math 703/704 homepage. In particular, note Claims 10 and 12 of this handout.

Fact.
Let $\{s_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Let $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.
Claim 10. There exists a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ of $\{s_n\}_{n=1}^{\infty}$ s.t. $\lim_{k\to\infty} s_{n_k} = \overline{\lim}_{n\to\infty} s_n \in \widehat{\mathbb{R}}$. Thus, if $\{s_n\}_{n=1}^{\infty}$ is bounded above, then it has a subsequence that converges to an element in $\mathbb{R}$.
Claim 12.

$$\overline{\lim_{n\to\infty}} \, s_n \;=\; \sup \left\{ \lim_{k\to\infty} s_{n_k} \;:\; \{s_{n_k}\}_{k=1}^{\infty} \text{ is a subsequence of } \{s_n\}_{n=1}^{\infty} \text{ s.t. } \lim_{k\to\infty} s_{n_k} \in \widehat{\mathbb{R}} \right\} .$$

Lemma 0.1. *Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of nonnegative real numbers. Then*

$$\overline{\lim_{n\to\infty}} \, (a_n)^2 \;=\; \left(\overline{\lim_{n\to\infty}} \, a_n \right)^2 .$$

Proof. LTGBG. Since $\{a_n\}_{n=1}^{\infty}$, and thus also $\left\{(a_n)^2\right\}_{n=1}^{\infty}$, are bounded above, the $\overline{\lim}_{n\to\infty} a_n$ and $\overline{\lim}_{n\to\infty} (a_n)^2$ are (finite) real numbers.
$\boxed{\leq}$. By Claim 10, there is a subsequence $\left\{(a_{n_k})^2\right\}_{k=1}^{\infty}$ of $\left\{(a_n)^2\right\}_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty} (a_{n_k})^2 \;=\; \overline{\lim_{n\to\infty}} \, (a_n)^2 .$$

Since the a_n's are nonnegative, $\overline{\lim}_{n\to\infty} a_n \geq 0$. Thus

$$\lim_{k\to\infty} a_{n_k} \;=\; \sqrt{\overline{\lim_{n\to\infty}} \, (a_n)^2} .$$

So by Claim 12, $\sqrt{\overline{\lim}_{n\to\infty} (a_n)^2} \;\leq\; \overline{\lim}_{n\to\infty} a_n$.
$\boxed{\geq}$. By Claim 10, $\exists$ a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ such that $\lim_{k\to\infty} a_{n_k} = \overline{\lim}_{n\to\infty} a_n$. But then $\lim_{k\to\infty} (a_{n_k})^2 = \left(\overline{\lim}_{n\to\infty} a_n\right)^2$. So by Claim 12, $\left(\overline{\lim}_{n\to\infty} a_n\right)^2 \leq \overline{\lim}_{n\to\infty} (a_n)^2$. $\square$

Now go back to Homework 1 ([K, Chapter 1, Section 11, #4) and see if you can find an easier proof than the proof that 16 of the 17 students gave. I tex-ed up the harder proof since this is how most students approached the problem.

Homework 32

Prove that $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges for each $z \in \mathbb{C}$ with $|z| = 1$ except $z = 1$.

You may use, without proving, the below *Summation by parts* Lemma 0.2.

Lemma 0.2. *Let $\{a_n\}_{n=1}^{N}$ and $\{b_n\}_{n=1}^{N}$ be finite sequences of complex numbers. Let $B_k = \sum_{l=1}^{k} b_l$. Then for $N > M > 1$*

$$\sum_{n=M}^{N} a_n b_n \;=\; a_N B_N \;-\; a_M B_{M-1} \;-\; \sum_{n=M}^{N-1} \left(a_{n+1} - a_n\right) B_n \;.$$

Hint for proof. Substitute $b_n = B_n - B_{n-1}$ in the sum on the left. □

Homework 33

Let $B_1(0)$ be the open unit disk in $\mathbb{C}$, i.e. $B_1(0) := \{z \in \mathbb{C} \colon |z| < 1\}$. Show that for $z \in B_1(0)$

$$\text{Log}\,(1 - z) \;=\; \sum_{n=1}^{\infty} -\frac{z^n}{n}$$

by proving that both sides are holomorphic on $B_1(0)$, agree at $z = 0$, and have the same derivative on $B_1(0)$.

Hint. Proposition I.4.10 from class notes and also [A, Theorem 3.1.2].

Homework 34

Let γ be the join of the three line segments $[1 - i, 1 + i]$ and $[1 + i, -1 + i]$ and $[-1 + i, -1 - i]$. Evaluate $\int_\gamma \frac{dz}{z}$ by using an appropriate branch of $\log z$.

Hint. See [A, Theorem 3.1.2].

Homework 35

Compute

$$\int_0^{2\pi} e^{\cos t} \left[\cos\left(t + \sin t\right)\right] \, dt \qquad \text{and} \qquad \int_0^{2\pi} e^{\cos t} \left[\sin\left(t + \sin t\right)\right] \, dt$$

by computing $\int_\gamma e^z \, dz$ where $\gamma \colon [0, 2\pi] \to \mathbb{C}$ is given by $\gamma(t) := e^{it}$.

Homework 36

Let $\alpha \in \mathbb{C}$ with $|\alpha| \neq 1$.
(a) Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(\theta) := e^{i\theta}$. Using Cauchy's Integral Formula (CIF) for starlike sets, calculate the following integral.

$$\int_\gamma \frac{dz}{(z-\alpha)\left(z-\frac{1}{\alpha}\right)} = \begin{cases} \boxed{} & \text{if } |\alpha| < 1 \\ \boxed{} & \text{if } |\alpha| > 1 \ . \end{cases}$$

When applying CIF for starlike sets, following the notation in the Script of this theorem [Thm. II.2.14, p. 21], clearly indicate what is your: open starlike set G, function $f\colon G \to \mathbb{C}$, and $z_0 \in G \setminus \gamma^*$.
(b) Using part (a), calculate the following integral.

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha\cos\theta + \alpha^2} = \begin{cases} \boxed{} & \text{if } |\alpha| < 1 \\ \boxed{} & \text{if } |\alpha| > 1 \ . \end{cases}$$

Hint. $e^{i\theta} + e^{-i\theta} = 2\cos\theta$.

Homework 37

Evaluate (without parametrizing the curve γ, but rather by using Cauchy's Integral Theorem)

$$\int_\gamma \frac{dz}{1+z^2}$$

for the following $\gamma\colon [0, 2\pi] \to \mathbb{C}$.

1. $\gamma(t) := 1 + e^{it}$
2. $\gamma(t) := -i + e^{it}$
3. $\gamma(t) := 2e^{it}$
4. $\gamma(t) := 3i + 3e^{it}$

Homework 38

Problem Source: Quals 1998.
Fix $a \in \mathbb{C}$ and $r > 0$ and let

$$\begin{aligned} B_r(a) &:= \{z \in \mathbb{C} \colon |z-a| < r\} \\ \overline{B_r(a)} &:= \{z \in \mathbb{C} \colon |z-a| \le r\} \\ \partial B_r(a) &:= \{z \in \mathbb{C} \colon |z-a| = r\} \ . \end{aligned}$$

Let G be an open set of $\mathbb{C}$ that contains $\overline{B_r(a)}$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on G such that $\{f_n\}_{n=1}^{\infty}$ converges to the zero function uniformly on $\partial B_r(a)$. Show that $\lim_{n\to\infty} f_n(z) = 0$ for each $z \in B_r(a)$.

Hint. Use Cauchy's Integral Formula.

Remark. $\{f_n\}_{n=1}^{\infty}$ converges to the zero function uniformly on $\partial B_r(a)$ means that

$$\lim_{n\to\infty} \sup_{z \in \partial B_r(a)} |f_n(z)| = 0 \ .$$

Homework 39

Problem Source: Quals 1995.
Let $f \in H(\mathbb{C})$ satifsy, for some constants $A, B \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$|f(z)| \ \le \ A\,|z|^k + B$$

for each $z \in \mathbb{C}$. Prove that f is a polynomial.

Homework 40

Problem Source: Quals 1995.
Let G be an open connected subset of $\mathbb{C}$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence from $H(G)$ and $f \colon G \to \mathbb{C}$ be a function satisying

$$\begin{gathered} \text{for each compact subset } K \text{ of } G, \\ \text{the functions } \{f_n|_K\}_{n=1}^{\infty} \text{ converge uniformly on } K \text{ to } f|_K \ . \end{gathered} \tag{40.1}$$

Show that $f \in H(G)$.

Homework 41

Problem Source: Quals 2000.
Let f be an entire function such that, for some $M > 0$,

$$|f(z)| \leq M e^{\operatorname{Re} z} \qquad \forall z \in \mathbb{C} .$$

Show that there exists $K \in \mathbb{C}$ such that $f(z) = K e^z$.

Homework 42

Problem Source: Quals 1999.
Show that

$$\int_0^{\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta \;=\; \pi \; .$$

Hint. Consider $\int_\gamma \frac{e^z}{z}\, dz$ where $\gamma \colon [0, 2\pi] \to \mathbb{C}$ is given by $\gamma(\theta) = e^{i\theta}$.

Homework 43

Problem Source: Quals 1997.
Let γ^* be as in Figure 1 below. Find

$$\int_\gamma \frac{1 + z + z^3}{z(z-1)(z-i)}\, dz \; .$$

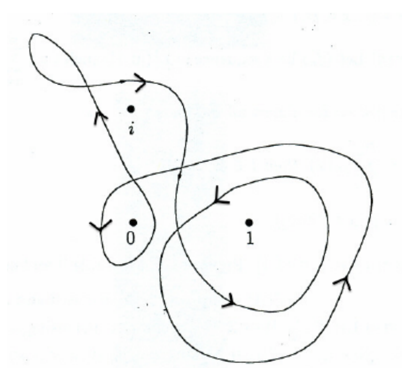

FIGURE 1