$\boxed{\# 13}$ Let (X,d) and (Y,ρ) be metric spaces.

Consider a function $f: X \to Y$.

Recall Prop 2.25 f is continuous at $x \in X$

$\Leftrightarrow$ if $\{x_n\}_{n=1}^{\infty} \subseteq X$ and $x_n \to x$, then $f(x_n) \to f(x)$.

Def f preserves convergent sequences

$\Leftrightarrow$ if $\{x_n\}_{n=1}^{\infty} \subseteq X$ converges, say $x_n \to x$, then $f(x_n) \to f(x)$.

So a Cor to Prop 2.25: f is continuous on $X \Leftrightarrow f$ preserves convergent sequences

Def f preserves Cauchy sequences

$\Leftrightarrow$ if $\{x_n\}_{n=1}^{\infty} \subseteq X$ is Cauchy, then $\{f(x_n)\}_{n=1}^{\infty} \subseteq Y$ is Cauchy.

$\boxed{13.1}$ Prove or give a counterexample.

f preserves convergent sequences $\Leftrightarrow$

if $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in X

then $\{f(x_n)\}_{n=1}^{\infty}$ is a convergent sequence in Y.

$\boxed{13.2}$ Show that a convergent sequence $\{x_n\}_{n=1}^{\infty}$ in X is Cauchy.

$\boxed{13.3}$ Give an example of a metric space (X,d) and a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ from X that does not converge. (Enough to quote previous homework problem.)

$\boxed{13.4}$ Give an example of a continuous $f: X \to Y$ that does not preserve Cauchy sequences.

$\boxed{13.5}$ Show that if f is uniformly continuous, then f preserves Cauchy sequences.

$\boxed{13.6}$ Prove or give a counterexample.

If f preserves Cauchy sequences, then f is uniformly continuous.

$\boxed{13.7}$ Show the following.

f is uniformly continuous $\Longleftrightarrow$

if $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are sequences from X such that $d(x_n, z_n) \to 0$

then $\rho(f(x_n), f(z_n)) \to 0$.

 # 14 Let (X, d) be a metric space.

Loosely speaking, show that a Cauchy sequence from X with a convergent subsequence converges.

Precisely, show the following.

Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence such that a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converges to some point $x \in X$.

Show that $\{x_n\}$ converges to x.

 # 15 Let (X, d) be a metric space and $\emptyset \neq A \subset X$.

Consider the metric space (A, d_A) where $d_A := d|_{A \times A}$.

15.1 Show that if A is complete, then A is closed in X.

15.2 Show that if A is totally bounded, then A is bounded.

 # 16 Let $\mathcal{U}$ be an open subset of $\mathbb{R}$.

Show that $\exists$ a finite or countably infinite indexing set A and intervals $\{I_j\}_{j \in A}$ s.t.

(1) I_j is open $\forall\, j \in A$

(2) $I_{j_1} \cap I_{j_2} = \emptyset$ if $j_1 \neq j_2$

(3) $\bigcup_{j \in A} I_j = \mathcal{U}$.