Prof. Girardi	Math 703	Fall 2012	11.26.12	Exam 1

MARK BOX		
PROBLEM	POINTS	
0	40	
1	20	
2	20	
3	20	
4	20	
5	20	
%	100	

NAME: ______________________________

INSTRUCTIONS:

(1) Do Problem 0. Do **3** of the problems: 1, 2, 3, 4, and 5.
(2) Use complete sentences.
(3) When asked to *show*, this really means *carefully show*. Write out details.
(4) Write your solutions on the provided unlined or lined homemade Blue Book paper. You do not need to recopy the statement of the problem (just use LTGBG).
- Start each new problem on a new sheet of paper.
- Write on only one side of a sheet of paper.
- On the top of each page put the problem number and your (first is enough) name.

(5) When finished:
- put your pages in order with this exam paper on top
- staple them together
- in the MARK BOX above, circle the problem that you chose to do.

Then hand in your completed homemade Blue Book.

(6) The MARK BOX indicates the problems along with their points. This test is copied 2-sided.
(7) You may **not** use any electronic devices, books, or personal notes.

Problem Source:

(1) [K]. Prop. 2.16, Prop. 2.19, and Prop. 2.30.
(2) [K]. Prop. 2.41.
(3) Homework 13.
(4) [K]. Prop. 2.47.
(5) [K]. Prop. 2.49.

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.
As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.
Furthermore, I have not only read but will also follow the above Instructions.

Signature : ______________________________

Throughout this exam, (X, d) and (Y, ρ) are metric spaces.

0a. Let:

$$f\colon X \to Y \text{ be a function} \quad \text{and} \quad \{x_n\}_{n=1}^{\infty} \text{ be a sequence in } X \quad \text{and} \quad A, D \subset X \ .$$

Carefully define the following concepts. (ps: these definitions will be used on this exam)

(1) $\{x_n\}_{n=1}^{\infty}$ converges to $x \in X$.

(2) $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

(3) f is continuous on X (using an ε and not using inverse images of open sets).

(4) f is uniformly continuous on X.

(5) f preserves Cauchy sequence.

(6) X is compact (via open coverings).

(7) A is a compact subset of X (via sequences, not open coverings, ok, this is really a theorem but can be taken to be a definition).

(8) X is complete.

(9) X is connected.

(10) D is dense in X.

0b. State and prove one of your more favorite results (Thm., Prop., Homework Problem, etc., but not one of the Problems 1–5 below) from the Real Analysis portion of this course.

1a. Let $\emptyset \neq A \subset X$. Define $f\colon X \to \mathbb{R}$ by

$$f(x) \ := \ D(x, A) \ \overset{\text{i.e.}}{=} \ \inf_{a \in A} d(x, a) \ .$$

Show that f is continuous on X.

1b. Let $\emptyset \neq A \subset X$. Denote the closure of A by $\overline{A}$. Show that $\overline{A} \ = \ \{x \in X \colon D(x, A) = 0\}$.

1c. Let E and F be nonempty disjoint closed subsets of X. Show that there is a continuous function $f\colon X \to [0, 1]$ such that f is zero exactly on E and f is 1 exactly on F.

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2. Let X be compact and $f\colon X \to Y$ be a continuous function. Show that f is uniformly continuous.

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3. Consider a function $f\colon X \to Y$.

3a. Prove or give a counterexample. If f preserves Cauchy sequences, then f is uniformly continuous.

3b. Let X be compact and Y be complete. Let f preserves Cauchy sequences.
Show that f is uniformly continuous.

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4. Let D be a dense subset of X and Y be complete. Let $f\colon D \to Y$ be uniformly continuous. Show

(a) f extends uniquely to a continuous function $F\colon X \to Y$

(b) the unique extension F from (a) is uniformly continuous.

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5. Let X be connected and $f\colon X \to Y$ be continuous. Show that $f(X)$ is connected.
