It follows that for large values of r the probability of *no* match between 2 London teams is approximately $e^{-1/2}$.

The formula is in fact very nearly correct for quite small values of r. For example, if there are 4 London teams out of 16 (as might happen in the fifth round), the actual probability is 8/13, which is 0·6154, while $e^{-1/2}$ is 0·6065 (both to 4 places).

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2742. A mean value theorem

The following problem (a simple consequence of Rolle's theorem) appeared in a recent University examination paper:

The function $g(x)$ is continuous in $a \leqslant x \leqslant b$, and

$$g(a) = 0, \qquad \int_a^b g(t)\,dt = 0. \qquad \text{......(1)}$$

By considering the function $\varphi(x)$ defined by the relations

$$\varphi(a) = 0, \qquad \varphi(x) = \frac{1}{x-a}\int_a^x g(t)\,dt \qquad (a < x \leqslant b),$$

prove that there is at least one ξ in $a < \xi < b$ such that

$$g(\xi) = \frac{1}{\xi - a}\int_a^\xi g(t)\,dt.$$

It is well known that the mean value φ of a function g is in general less irregular in its behaviour than g itself, and the result above merely states the intuitively obvious fact that if the curves $y = g(x)$, $y = \varphi(x)$ start together, and if the former oscillates, then the two will meet at some subsequent point.

We may ask whether the second condition in (1) may be replaced by a simpler condition, for example, by the condition $g(b) = 0$.* The answer is affirmative, and the result in this more general form can be deduced without difficulty from the theorem that a continuous function attains all values between its bounds.

By writing $f(x) = \int_a^x g(t)\,dt$, we obtain an equivalent form of the theorem which might be considered to have independent interest:

If $f(x)$ is differentiable and $f'(x)$ is continuous in $a \leqslant x \leqslant b$, and $f'(a) = f'(b) = 0$, then there is at least one ξ in $a < \xi < b$ such that

$$\frac{f(\xi) - f(a)}{\xi - a} = f'(\xi).$$

This last theorem has a simple geometrical interpretation, namely that if the curve $y = f(x)$ has a continuously turning tangent in $a \leqslant x \leqslant b$, and if the tangents at $x = a$ and $x = b$ are parallel, then there is an intermediate point ξ such that the tangent there passes through the point a. It is easily seen that the condition that the tangents at a, b be parallel is the natural one here, and in this form the theorem is even more obvious on intuitive grounds than is the integral form. There is, however, one last extension which we can make—we can omit the hypothesis that f' is continuous. The resulting theorem is a good deal less obvious, and the proof is perhaps more difficult. We have in fact the

* The second condition in (1) implies that g must vanish somewhere between a and b.

THEOREM. *If $f(x)$ is differentiable in $a \leqslant x \leqslant b$, and $f'(a) = f'(b)$, then there exists a point ξ in $a < \xi < b$ such that the tangent to the curve $y = f(x)$ at the point ξ passes through the point a, i.e. such that*

$$\frac{f(\xi) - f(a)}{\xi - a} = f'(\xi).$$

We may suppose that $f'(a) = f'(b) = 0$, for if this is not the case we work with $f(x) - xf'(a)$. Consider now the function ψ defined by the relations

$$\psi(a) = f'(a) = 0, \qquad \psi(x) = \frac{f(x) - f(a)}{x - a} \qquad (a < x \leqslant b).$$

Evidently ψ is continuous in $a \leqslant x \leqslant b$ and differentiable in $a < x \leqslant b$, and

$$\psi'(x) = -\frac{f(x) - f(a)}{(x - a)^2} + \frac{f'(x)}{x - a} \qquad (a < x \leqslant b).$$

It is therefore sufficient to prove that there is some point ξ in $a < \xi < b$ such that $\psi'(\xi) = 0$.

This is an immediate consequence of Rolle's theorem if $\psi(b) = 0$. Suppose then that $\psi(b) > 0$, so that $\psi'(b) = -\psi(b)/(b - a) < 0$. Then there exists x_1 in $a < x_1 < b$ such that $\psi(x_1) > \psi(b)$. Since ψ is continuous in $a \leqslant x \leqslant x_1$ and $\psi(a) < \psi(b) < \psi(x_1)$, there is a point x_2 in $a < x_2 < x_1$ such that $\psi(x_2) = \psi(b)$, and the required result now follows from Rolle's theorem applied to the function ψ in the interval (x_2, b). A similar argument applies if $\psi(b) < 0$, and this completes the proof.

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2743. An inequality

In connexion with Mr C. V. Durell's letter (*Math. Gaz.* 50 (1956), 266), some further information about the alleged inequality

$$f_n(x) \equiv \frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \dots + \frac{x_{n-1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2} \geqslant \tfrac{1}{2}n, \quad \text{......(1)}$$

where $x_i > 0$ for $i = 1, 2, \dots, n$, can be found in the *American Mathematical Monthly* 63 (1956), 191–192. Here Lighthill's example of the falsity of (1) for $n = 20$ is given by F. H. Northover, and it is stated that (1) has been proved to be true for $n = 5$ by C. R. Phelps.

Let $\mu(n)$ be the greatest lower bound of $f_n(x)$ for $x_i > 0$ $(i = 1, 2, \dots, n)$, so that $\mu(n) = \frac{1}{2}n$ for $n \leqslant 5$.

By taking n even and choosing

$$x_{2r-1} = 1 + \varepsilon a_r, \qquad x_{2r} = \varepsilon a_{r-1} \qquad (1 \leqslant r \leqslant m = \tfrac{1}{2}n),$$

with positive ε, a_r and $a_0 = a_m$, Professor Lighthill has shown* that, more generally,

$$\mu(n) < \tfrac{1}{2}n \qquad (n > 12,\ n \text{ even}). \quad \text{......(2)}$$

In the particular case $n = 16$, I have made a rough numerical calculation based on his method and find that

$$\mu(16) < 16c, \qquad \text{where } c = \tfrac{1}{2} - 7 \times 10^{-8}. \quad \text{......(3)}$$

The following additional results may be of interest.

* Private communication.