Notation 1.1. Throughout this presentation:

- \((a, b) = I \subseteq \mathbb{R}\) where \(a, b \in \mathbb{R} \cup \{\pm \infty\}\)
- \(\varphi: I \to \mathbb{R}\) is a function
- \(p \in (1, \infty)\) and its conjugate exponent \(p' \in (1, \infty)\) is defined by \(\frac{1}{p} + \frac{1}{p'} = 1\).

Definition 1.2. \(\varphi: I \to \mathbb{R}\) is convex \(\iff\)

\[
x, y \in I, \ t \in (0, 1) \implies \varphi(tx + (1 - t)y) \leq t\varphi(x) + (1 - t)\varphi(y).
\] (1.1)

Picture.

\[\begin{array}{cccccc}
a & x & tx + (1 - t)y & y & b\
\end{array}\]

Date: 7 June 2001.
2. Holder’s Inequality

Lemma 2.1. Let \( \varphi : I \to \mathbb{R} \) be convex. Then:

\[
x_i \in I , \quad t_i \in (0, 1) , \quad \sum_{i=1}^{n} t_i = 1 \quad \implies \quad \varphi \left( \sum_{i=1}^{n} t_i x_i \right) \leq \sum_{i=1}^{n} t_i \varphi (x_i)
\]  

(2.1)

\[
x_i \in I , \quad t_i \in (0, 1) \quad \implies \quad \varphi \left( \frac{\sum_{i=1}^{n} t_i x_i}{\sum_{i=1}^{n} t_i} \right) \leq \frac{\sum_{i=1}^{n} t_i \varphi (x_i)}{\sum_{i=1}^{n} t_i}.
\]  

(2.2)

Proof. (2.1): Use (1.1) and induction. (2.2): Let \( \tilde{t}_i = \left[ \sum_{j=1}^{n} t_j \right]^{-1} t_i \) and apply (2.1).

Recall. Geometric-Arithmetic Mean Inequality: \( \text{GM} \leq \text{AM} \)

\[
x_i \geq 0 , \quad n \in \mathbb{N} \quad \implies \quad \left( \prod_{i=1}^{n} x_i \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i.
\]

Using convex functions, we can generalize the GM-AM inequality.

Proposition 2.2. Generalized GM-AM inequality:

\[
x_i \geq 0 , \quad t_i \in (0, 1) , \quad \sum_{i=1}^{n} t_i = 1 \quad \implies \quad \prod_{i=1}^{n} x_i^{t_i} \leq \sum_{i=1}^{n} t_i x_i.
\]  

(2.3)

Proof. Let \( \varphi : (0, \infty) \to \mathbb{R} \) be \( \varphi (x) = -\ln x \). Then

\[
\varphi \text{ is convex} \quad \implies \quad -\ln \left( \sum_{i=1}^{n} t_i x_i \right) \leq \sum_{i=1}^{n} t_i \left( -\ln x_i \right)
\]

\[
\implies \sum_{i=1}^{n} \ln \left( x_i^{t_i} \right) \leq \ln \left( \sum_{i=1}^{n} t_i x_i \right).
\]  

(2.4)

Now exponentiate both sides of (2.4).

An immediate corollary follows now.

Corollary 2.3. Young’s Inequality:

\[
x_i \geq 0 , \quad 1 < p < \infty \quad \implies \quad x_1 \cdot x_2 \leq \frac{x_1^{p}}{p} + \frac{x_2^{p'}}{p'}.
\]  

(2.5)
Proof. Apply (2.3) to
\[ x_1 \cdot x_2 \equiv (x_1^p)^{1/p} \cdot (x_2^{p'})^{1/p'} . \]

Recall. For a sequence \( \{x_i\}_{i=1}^n = \{x_1, \ldots, x_n\} \) from \( \mathbb{R} \):

\[ \|\{x_i\}_{i=1}^n\|_{\ell_p} = \left[ \sum_{i=1}^n |x_i|^p \right]^{1/p} . \]

**Theorem 2.4.** Holder’s Inequality in \( \ell_p \): For sequences \( \{x_i\}_{i=1}^n \) and \( \{y_i\}_{i=1}^n \) from \( \mathbb{R} \):

\[ \|\{x_i \cdot y_i\}_{i=1}^n\|_{\ell_1} \leq \|\{x_i\}_{i=1}^n\|_{\ell_p} \cdot \|\{y_i\}_{i=1}^n\|_{\ell_{p'}} \]

that is

\[ \sum_{i=1}^n |x_i y_i| \leq \left[ \sum_{i=1}^n |x_i|^p \right]^{1/p} \cdot \left[ \sum_{i=1}^n |y_i|^{p'} \right]^{1/p'} . \quad (2.6) \]

When \( p = 2 = p' \), note that (2.6) is just the Cauchy-Schwarz Inequality.

**Proof.** WLOG: Neither \( \|\{x_i\}_{i=1}^n\|_{\ell_p} \) nor \( \|\{y_i\}_{i=1}^n\|_{\ell_{p'}} \) is 0. WLOG: \( \|\{x_i\}_{i=1}^n\|_{\ell_p} = 1 = \|\{y_i\}_{i=1}^n\|_{\ell_{p'}} \)

for if not then:

\[ \tilde{x}_i := \frac{x_i}{\|\{x_j\}_{j=1}^n\|_{\ell_p}} \implies \|\{\tilde{x}_i\}_{i=1}^n\|_{\ell_p} = 1 \quad , \quad \tilde{y}_i := \frac{y_i}{\|\{y_j\}_{j=1}^n\|_{\ell_{p'}}} \implies \|\{\tilde{y}_i\}_{i=1}^n\|_{\ell_{p'}} = 1 . \]

By Young’s Inequality

\[ \sum_{i=1}^n |x_i y_i| \leq \sum_{i=1}^n \left[ \frac{|x_i|^p}{p} + \frac{|y_i|^{p'}}{p'} \right] = \frac{1}{p} \|\{x_i\}_{i=1}^n\|_{\ell_p}^p + \frac{1}{p'} \|\{y_i\}_{i=1}^n\|_{\ell_{p'}}^{p'} = 1 . \]

Here are two exercises of Generalized Holder’s Inequalities in \( \ell_p \).

**Exercise.** For sequences \( \{x_i^j\}_{i=1}^n \) from \( \mathbb{R} \) and \( \sum_{j=1}^k \frac{1}{p_j} = 1 \)

\[ \left\| \left\{ \prod_{j=1}^k x_i^j \right\}_{i=1}^n \right\|_{\ell_1} \leq \prod_{j=1}^k \left\| \{x_i^j\}_{i=1}^n \right\|_{\ell_{p_j}} . \]

3
Exercise. For sequences \( \{x_i\}_{i=1}^n \) and \( \{y_i\}_{i=1}^n \) from \( \mathbb{R} \) and \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} \)

\[
\| \{x_i \cdot y_i\}_{i=1}^n \|_{\ell_{p_3}} \leq \| \{x_i\}_{i=1}^n \|_{\ell_{p_1}} \cdot \| \{y_i\}_{i=1}^n \|_{\ell_{p_2}}.
\]

Hint: \( \frac{1}{p_1/p_3} + \frac{1}{p_2/p_3} = 1 \).

Recall. For a nice function \( f : I \rightarrow \mathbb{R} \),

\[
\|f\|_{L_p} = \left[ \int_I |f(x)|^p \, dx \right]^{1/p}.
\]

**Theorem 2.5.** Holder’s Inequality in \( L_p \): For nice functions \( f, g : I \rightarrow \mathbb{R} \),

\[
\|f \cdot g\|_{L_1} \leq \|f\|_{L_p} \cdot \|g\|_{L_{p'}}
\]

that is,

\[
\int_I |f(x)g(x)| \, dx \leq \left[ \int_I |f(x)|^p \, dx \right]^{1/p} \cdot \left[ \int_I |g(x)|^{p'} \, dx \right]^{1/p'}
\]

**Proof.** WLOG: Neither \( \|f\|_{L_p} \) nor \( \|g\|_{L_{p'}} \) is 0 or \( \infty \). WLOG: \( \|f\|_{L_p} = 1 = \|g\|_{L_{p'}} \) for if not so:

\[
\tilde{f} := \frac{f}{\|f\|_{L_p}} \quad \Rightarrow \quad \|\tilde{f}\|_{L_p} = 1, \quad \tilde{g} := \frac{g}{\|g\|_{L_{p'}}} \quad \Rightarrow \quad \|\tilde{g}\|_{L_{p'}} = 1.
\]

By Young’s Inequality

\[
\int_I |f(x)g(x)| \, dx \leq \int_I \left[ \frac{|f(x)|^p}{p} + \frac{|g(x)|^{p'}}{p'} \right] = \frac{1}{p} \|f\|_{L_p}^p + \frac{1}{p'} \|g\|_{L_{p'}}^{p'} = 1.
\]

Here are two exercises of Generalized Holder’s Inequalities in \( L_p \).

**Exercise.** For nice functions \( f_j : I \rightarrow \mathbb{R} \) and \( \sum_{j=1}^k \frac{1}{p_j} = 1 \)

\[
\left\| \prod_{j=1}^k f_j \right\|_{L_1} \leq \prod_{j=1}^k \|f_j\|_{L_{p_j}}.
\]

**Exercise.** For nice functions \( f, g : I \rightarrow \mathbb{R} \) and \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} \)

\[
\|f \cdot g\|_{L_{p_3}} \leq \|f\|_{L_{p_1}} \cdot \|g\|_{L_{p_2}}.
\]
Lemma 3.1. Let $\varphi: I \to \mathbb{R}$ be convex. Let $x, y, z \in I$ with $x < y < z$. Then

$$
\frac{\varphi(y) - \varphi(x)}{y - x} \leq \frac{\varphi(z) - \varphi(x)}{z - x} \leq \frac{\varphi(z) - \varphi(y)}{z - y} .
$$

(3.1)

Proof. Key idea:

$$
y = tx + (1 - t)z = \frac{z - y}{z - x} x + \frac{y - x}{z - x} z \implies \varphi(y) \leq t\varphi(x) + (1 - t)\varphi(z) .
$$

Think of what this says in the picture; the needed algebra then follows easily.
Proposition 3.2. Let \( \varphi : I \to \mathbb{R} \) be convex and \( x_0 \in I \). Then

1. \( \varphi \) is continuous at \( x_0 \)
2. \( \varphi'_-(x_0) := \lim_{x_l \to x_0^-} \frac{\varphi(x_l) - \varphi(x_0)}{x_l - x_0} \) exists
3. \( \varphi'_+(x_0) := \lim_{x_r \to x_0^+} \frac{\varphi(x_r) - \varphi(x_0)}{x_r - x_0} \) exists
4. \( \varphi'_-(x_0) \leq \varphi'_+(x_0) \).

Proof. (1): Find: \( a < a_1 < a_2 < x_0 < b_1 < b < b_2 \). If \( x \in [a_2, b_2] \setminus \{x_0\} \), then

\[
\begin{array}{ccccccc}
\text{←} & a & a_1 & a_2 & x & x_0 & x & b_2 & b_1 & \text{→} \\
\end{array}
\]

by (3.1)

\[
A := \frac{\varphi(a_2) - \varphi(a_1)}{a_2 - a_1} \leq \frac{\varphi(x_0) - \varphi(x)}{x_0 - x} \leq \frac{\varphi(b_1) - \varphi(b_2)}{b_1 - b_2} := B
\]

and so

\[
\left| \frac{\varphi(x_0) - \varphi(x)}{x_0 - x} \right| \leq \max (|A|, |B|).
\]

(2)–(4): Consider

\[
\begin{array}{ccccccc}
\text{←} & a & x_l & \tilde{x}_l & x_0 & \tilde{x}_r & x_r & b \text{→} \\
\end{array}
\]

By (3.1)

\[
\frac{\varphi(x_0) - \varphi(x_l)}{x_0 - x_l} \leq \frac{\varphi(x_0) - \varphi(\tilde{x}_l)}{x_0 - \tilde{x}_l} \leq \frac{\varphi(\tilde{x}_r) - \varphi(x_0)}{\tilde{x}_r - x_0} \leq \frac{\varphi(x_r) - \varphi(x_0)}{x_r - x_0}
\]

and so

\[
\lim_{x_l \to x_0} \frac{\varphi(x_0) - \varphi(x_l)}{x_0 - x_l} \leq \varphi'_-(x_0) \leq \lim_{x_r \to x_0} \frac{\varphi(x_r) - \varphi(x_0)}{x_r - x_0}.
\]

(3.2)
Observation 3.3. Let \( \varphi : I \to \mathbb{R} \) be convex and \( x_0 \in I \). Let
\[
\varphi'_-(x_0) \leq m \leq \varphi'_+(x_0).
\]
Consider the line
\[
l(x) = m(x - x_0) + \varphi(x_0)
\]
through the point \((x_0, \varphi(x_0))\). Then
\[
l(x) \leq \varphi(x) \quad \forall x \in I.
\] (3.3)

A line \( y = l(x) \) through the point \((x_0, \varphi(x_0))\) that satisfies (3.3) is called a supporting line of \( y = \varphi(x) \) at \( x_0 \). Draw yourself a picture to see the choice of terminology here. Thus Observation 3.3 says that convex functions always have supporting lines.

Picture.

\[
\begin{array}{cccccc}
& & & & & \\
\vdots & & & & & \\
a & x_l & x_0 & x_r & b & \\
\end{array}
\]

Proof. By (3.2)
\[
\frac{\varphi(x_0) - \varphi(x_l)}{x_0 - x_l} \leq m \leq \frac{\varphi(x_r) - \varphi(x_0)}{x_r - x_0}.
\]

Thus
\[
m(x - x_0) \leq \varphi(x) - \varphi(x_0) \quad \forall x \in I \setminus \{x_0\}.
\]
**Theorem 3.4.** Jensen’s Inequality: Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be convex and \( f : I \to \mathbb{R} \) be integrable.

If \( I = (0, 1) \), then

\[
\varphi \left( \int_I f(x) \, dx \right) \leq \int_I \varphi (f(x)) \, dx \tag{3.4}
\]

If \( I = (a, b) \) has finite length, then

\[
\varphi \left( \frac{\int_I f(x) \, dx}{b-a} \right) \leq \frac{\int_I \varphi (f(x)) \, dx}{b-a} . \tag{3.5}
\]

**Remark.** Theorem 3.4 may be thought of as a continuous version of Lemma 2.1.

**Proof.** (3.4): Let \( \int_I f(x) \, dx = x_0 \) and \( \varphi'_-(x_0) \leq m \leq \varphi'_+(x_0) \). Then by Observation 3.3

\[
m(x-x_0) + \varphi(x_0) \leq \varphi(x) \quad \forall x \in \mathbb{R} .
\]

Thus

\[
m(f(x) - x_0) + \varphi(x_0) \leq \varphi(f(x)) \quad \forall x \in I . \tag{3.6}
\]

Now integrate both sides of (3.6) over \( I := (0, 1) \).

(3.5) follows applying (3.4) to

\[
g(x) := f (a + x (b-a)) : I \to \mathbb{R} .
\]