Thm All these approaches to the $\underset{\text{R.I.}}{\underset{\parallel}{\text{Riemann Integral}}}$ are equivalent for bounded functions.

Let $f:[a,b]\to\mathbb{R}$ be bounded.

Then TFAE :

(1) $f\in\mathcal{R}[a,b]$ in the sense of Def 3.1.3 / tagged partitions, i.e.

$(\exists L\in\mathbb{R})(\forall\varepsilon>0)(\exists\delta>0)\left[\dot{\mathcal{P}}\in\dot{\mathbb{P}}_{[a,b]} \text{ and } \|\dot{\mathcal{P}}\|<\delta \Rightarrow |S(f,\dot{\mathcal{P}})-L|<\varepsilon\right]$

(2) f satisfies the Cauchy Criteria for R.I., ie

$(\forall\varepsilon>0)(\exists\delta>0)\left[\dot{\mathcal{P}}_1,\dot{\mathcal{P}}_2\in\dot{\mathbb{P}}_{[a,b]} \;\&\; \|\dot{\mathcal{P}}_1\|<\delta \;\&\; \|\dot{\mathcal{P}}_2\|<\delta \Rightarrow |S(f,\dot{\mathcal{P}}_1)-S(f,\dot{\mathcal{P}}_2)|<\varepsilon\right]$

(3) $f\in\mathcal{R}[a,b]$ in the sense of upper & lower sums, i.e.

$$\underline{\int_a^b} f = \overline{\int_a^b} f \qquad (:=\tilde{L})$$

side remark Know $\underline{\int_a^b} f \le \overline{\int_a^b} f$

so $\underline{\int_a^b} f = \overline{\int_a^b} f \iff \left[(\forall\varepsilon>0)\left(\overline{\int_a^b} f - \underline{\int_a^b} f<\varepsilon\right)\right]$

(4) f satisfies Riemann's Integrability Criterion, ie.

$(\forall\varepsilon>0)(\exists\mathcal{P}\in\mathbb{P}_{[a,b]})\left[\mathcal{U}(\mathcal{P},f)-\mathcal{L}(\mathcal{P},f)<\varepsilon.\right]$

Futhermore, if $f\in\mathcal{R}[a,b]$, then [the L from (1) = the $\tilde{L}$ from (3)]

Pf. Let $f:[a,b]\to\mathbb{R}$ be bounded., say $\sup_{x\in[a,b]}|f(x)|:=B$.

$$(2) \iff (1) \underset{3.2.6}{\overset{\text{Thm}}{\Longleftrightarrow}} (3) \underset{3.2.7}{\overset{\text{Thm}}{\Longleftrightarrow}} (4).$$

↑ already did

Pf... in small steps to come.