WEIERSTRASS'S NON-DIFFERENTIABLE FUNCTION

BY

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CONTENTS

1. Introduction

1.1. It was proved by Weierstrass* that the function

$$f(x) = \sum a^n \cos b^n \pi x, \tag{1.11}$$

where b is an odd integer and

$$0 < a < 1, \tag{1.121}$$

$$ab > 1 + \frac{3}{2}\pi, \tag{1.122}$$

has no differential coefficient for any value of x. Weierstrass's result has been generalized very widely by a number of writers,† who have considered

* Weierstrass, *Abhandlungen aus der Functionenlehre*, p. 97 (see also P. du Bois-Reymond, *Versuch einer Classification der willkürlichen Functionen reeller Argumente nach ihren Änderungen in den kleinsten Intervallen*, Journal für Mathematik, vol. 79 (1875), pp. 21–37).

† I may refer in particular to

Darboux, *Mémoire sur les fonctions discontinues*, Annales de l'École Normale, ser. 2, vol. 4 (1875), pp. 57–112 (pp. 107–108), and vol. 8 (1879), pp. 195–202; Faber, *Einfaches Beispiel einer stetigen nirgends differentiirbaren Funktion*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 16 (1907), pp. 538–540; Landsberg, *Über Differentiirbarkeit stetiger Funktionen*, ibid., vol. 17 (1908), pp. 46–51; Lerch, *Über die Nichtdifferentiirbarkeit gewisser Functionen*, Journal für Mathematik, vol. 103 (1888), pp. 126–138;

and to

Bromwich, *Infinite Series*, pp. 490–491; Dini, *Grundlagen*, pp. 205 *et seq.*; Hobson, *Functions of a real variable*, pp. 620 *et seq.*

For a further discussion of certain points concerning Weierstrass's function in particular, see: Wiener, *Geometrische und analytische Untersuchung der Weierstrass'schen Function*, Journal für Mathematik, vol. 90 (1881), pp. 221–252.

I must confess that I have not been able to arrive at a proper understanding of all the contents of this paper.

functions of the general forms

$$C(x) = \sum a_n \cos b_n x, \tag{1.131}$$

$$S(x) = \sum a_n \sin b_n x, \tag{1.132}$$

where the a's and b's are positive, the series $\sum a_n$ is convergent, and the b's increase steadily and with more than a certain rapidity.

A study of the writings to which I have referred, and in particular of the parts of them which bear directly upon Weierstrass's function, soon shows that the last word has not yet been said upon the subject. In this paper I develop a new method for the discussion of this and similar questions, a method less elementary but considerably more powerful than those adopted hitherto. It would be easy to apply it in such a manner as to frame very general conditions for the non-differentiability of the series (1.13). I have not thought it worth while, however, to do this. The interest of my analysis lies, I think, in the method itself and in the results which it gives in a few particularly simple and interesting cases; and its greater power is quite sufficiently illustrated by its application to Weierstrass's classical example.

1.2. The known results concerning Weierstrass's cosine series are, so far as I am aware, as follows. Weierstrass gave the condition

$$ab > 1 + \frac{3}{2}\pi, \tag{1.122}$$

and the only direct improvement that I know on this is Bromwich's

$$ab > 1 + \frac{3}{2}\pi(1 - a). \tag{1.21}$$

These conditions forbid the existence of a differential coefficient *finite or infinite*. For the non-existence of a *finite* differential coefficient there are alternative conditions: Dini's

$$ab \geqq 1, \qquad ab^2 > 1 + 3\pi^2, \tag{1.221}$$

Lerch's

$$ab \geqq 1, \qquad ab^2 > 1 + \pi^2, \tag{1.222}$$

and finally Bromwich's

$$ab \geqq 1, \qquad ab^2 > 1 + \frac{3}{4}\pi^2(1 - a). \tag{1.223}$$

All these conditions presuppose that b is an odd integer. But Dini has also shown that if (1.122) is replaced by

$$ab > 1 + \frac{3}{2}\pi\frac{1 - a}{1 - 3a}, \tag{1.231}$$

or (1.221) by

$$ab \geqq 1, \qquad ab^2 > 1 + 15\pi^2 \frac{1-a}{5-21a}, \tag{1.232}$$

then this restriction may be removed. It is naturally presupposed in (1.231) that $a < \frac{1}{3}$ and in (1.232) that

$$a < \frac{5}{21}.$$

These conditions are all obviously artificial. It would be difficult to believe that any of them really correspond to any essential feature of the problem under discussion. They arise merely in consequence of the limitations of the methods employed. There is in fact only one condition which suggests itself naturally and seems obviously relevant, viz:

$$ab \geqq 1.* \tag{1.24}$$

1.3. The chief results which I prove here concerning Weierstrass's function, and the corresponding function defined by a series of sines, may be summarized as follows. In none of the results is b restricted to be an integer.

THEOREM 1.31. *Neither of the functions*

$$C(x) = \sum a^n \cos b^n \pi x, \qquad S(x) = \sum a^n \sin b^n \pi x,$$

where $0 < a < 1$, $b > 1$, *possesses a finite differential coefficient at any point in any case in which*

$$ab \geqq 1.$$

THEOREM 1.32. *Theorem* 1.31 *becomes untrue if the word "finite" is omitted.*

1.33. *If* $ab > 1$ *and so*

$$\xi = \frac{\log(1/a)}{\log b} < 1,$$

then each of the functions satisfies the condition

$$f(x+h) - f(x) = O(|h|^{\xi}),$$

for every value of x*; but neither of them satisfies*

$$f(x+h) - f(x) = o(|h|^{\xi}),$$

for any value of x.

* Hadamard (*La série de Taylor et son prolongement analytique*, p. 31, f. n.), referring to the sine-series, remarks: "Cette hypothèse ($ab > 1 + \frac{3}{2}\pi$) est celle que nécessite le raisonnement de Weierstrass. En réalité, il suffit que $ab > 1$"; but no proof of this assertion has ever been published, and I understand it as merely the expression of an opinion, the more so since the wording of the remark is ambiguous. It is not stated whether b is merely an integer (as in Hadamard's text), or an odd integer (as in Weierstrass's discussion of the cosine-series), nor whether the condition precludes the existence of any differential coefficient or only of a finite one. If the differential coefficient is not restricted to be finite, the assertion is, as we shall see later, untrue. If it is so restricted, then it is enough that $ab \geqq 1$.

In section 2 I prove these theorems on the assumption that b is integral. In section 3 I extend the results to the general case. In section 4 I give a simple example of a function, represented by an absolutely convergent Fourier series, which does not satisfy a "Lipschitz condition" of any order for any value of x; I introduce a short digression concerning a theorem of S. Bernstein; and I discuss the question of the differentiability of the function

$$f(x) = \sum \frac{\sin n^2 \pi x}{n^2}.$$

This function is of interest for historical reasons, as it was supposed not to be differentiable by Riemann and his pupils.* I prove here that $f(x)$, and indeed the function derived from $f(x)$ by replacing n^2 by n^α, where $\alpha < \frac{5}{2}$, has no finite differential coefficient for any *irrational* value of α. This result lies a good deal deeper than anything else in the paper. The proof which I give is, as it stands, simple, but it depends on previous results established by Mr. Littlewood and myself by reasoning of a highly transcendental character.

2. Weierstrass's function when b is an integer

2.1. I shall suppose throughout this section that b is integral, and I shall write $\pi x = \theta$, so that Weierstrass's series is a Fourier series in θ. I shall begin by proving a series of lemmas.

Suppose that $G(r, \theta)$ is a harmonic function, the real part of a power-series

$$\sum a_n z^n = \sum a_n r^n e^{ni\theta}$$

convergent when $r < 1$. Suppose further that $G(r, \theta)$ is continuous for $r \leqq 1$, and that

$$G(1, \theta) = g(\theta).$$

Lemma 2.11. *If*

$$g(\theta) - g(\theta_0) = o(|\theta - \theta_0|^\alpha),$$

where $0 < \alpha < 1$, *when* $\theta \to \theta_0$, *then*

$$\frac{\partial G(r, \theta_0)}{\partial \theta_0} = o\{(1 - r)^{-(1-\alpha)}\}$$

when $r \to 1$.

It is obvious that we may, without loss of generality, suppose $\theta_0 = 0$. We have

$$(2.111) \qquad G(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(u - \theta) + r^2} g(u)\, du,$$

when $r < 1$. Differentiating with respect to θ, and then putting $\theta = \theta_0 = 0$, we obtain

* See du Bois-Reymond's memoir quoted on p. 301 (p. 28).

$$\frac{\partial G}{\partial \theta_0} = \frac{r(1-r^2)}{\pi} \int_{-\pi}^{\pi} \frac{\sin u}{\Delta^2} g(u)\,du, \tag{2.112}$$

where

$$\Delta = 1 - 2r\cos u + r^2. \tag{2.113}$$

Let

$$g(u) - g(0) = \gamma(u). \tag{2.114}$$

Then we can choose δ so that

$$|\gamma(u)| < \epsilon |u|^{a} \tag{2.115}$$

for $-\delta \leqq u \leqq \delta$. Also

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{\sin u}{\Delta^2} g(u)\,du &= \int_{-\pi}^{\pi} \frac{\sin u}{\Delta^2} \gamma(u)\,du \\ &= \left(\int_{-\pi}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\pi} \right) \frac{\sin u}{\Delta^2} \gamma(u)\,du \\ &= J_1 + J_2 + J_3, \end{aligned} \tag{2.116}$$

say. Plainly

$$J_1 = O(1), \qquad J_3 = O(1), \tag{2.117}$$

when $r \to 1$. Also

$$\begin{aligned} |J_2| &< 2\epsilon \int_0^{\delta} \frac{u^{a+1}\,du}{\Delta^2} = 2\epsilon \int_0^{\delta} \frac{u^{a+1}\,du}{\{(1-r)^2 + 4r\sin^2 \tfrac{1}{2}u\}^2} \\ &< 32\epsilon \int_0^{\delta} \frac{u^{a+1}\,du}{\{(1-r)^2 + ru^2\}^2} {}^{*} < 32\epsilon \int_0^{\infty} \frac{u^{a+1}\,du}{\{(1-r)^2 + ru^2\}^2} \\ &= 32\epsilon \frac{(1-r)^{a-2}}{r^{1+\frac{1}{2}a}} \int_0^{\infty} \frac{w^{a+1}\,dw}{(1+w^2)^2}. \end{aligned} \tag{2.118}$$

From (2.112), (2.116), (2.117), and (2.118) it follows that

$$\left| \frac{\partial G}{\partial \theta_0} \right| < K\epsilon (1-r)^{-(1-a)}, \tag{2.119}$$

where K is a constant, for all values of r near enough to 1; and the lemma is therefore proved.

LEMMA 2.12. *If $g(\theta)$ possesses a finite differential coefficient $g'(\theta_0)$ for $\theta = \theta_0$, then*

$$\frac{\partial G}{\partial \theta_0} \to g'(\theta_0)$$

when $r \to 1$.

This is a known theorem, due to Fatou.† The principle of the proof does

* Since $1 - r > \frac{1}{2}(1-r)$ and $\sin \frac{1}{2}u \geqq \frac{1}{4}u$.

† *Séries trigonométriques et séries de Taylor,* Acta Mathematica, vol. 30 (1906), pp. 335–400.

not differ essentially from that of Lemma 2.11, and I need hardly repeat it.

LEMMA 2.13. *Suppose that* $f(y)$ *is a real or complex function of the real variable* y, *possessing a p-th differential coefficient* $f^{(p)}(y)$ *continuous throughout the interval* $0 < y \leqq y_0$. *Suppose further that* $\lambda \geqq 0$, *that*

$$f(y) = o(y^{-\lambda})$$

if $\lambda > 0$ *and*

$$f(y) = A + o(1)$$

if $\lambda = 0$, *and in either case that*

$$f^{(p)}(y) = O(y^{-p-\lambda}).$$

Then

$$f^{(q)}(y) = o(y^{-q-\lambda})$$

for $0 < q < p$.

This is a special case of a result proved by Mr. Littlewood and myself in 1912.*

LEMMA 2.14. *If* $\rho > 0$ *and*

$$f(y) = \sum b^{n\rho} e^{-b^n y}$$

then

$$f(y) = O(y^{-\rho})$$

as $y \to 0$.

For, if we write

$$e^{-y} = u, \qquad f(y) = \sum a_n u^n, \tag{2.141}$$

$$a_0 + a_1 + \cdots + a_n = s_n, \tag{2.142}$$

then

$$\begin{aligned} s_n &= 1 + b^\rho + b^{2\rho} + \cdots + b^{\nu\rho} \\ &= O(b^{\nu\rho}) = O(n^\rho) \end{aligned} \tag{2.143}$$

for

$$b^\nu \leqq n < b^{\nu+1}.$$

From (2.141), (2.142), and (2.143) it follows that

$$f(y) = (1-u)\sum s_n u^n = O\{(1-u)\sum n^\rho u^n\} = O\{(1-u)^{-\rho}\} = O(y^{-\rho}).$$

LEMMA 2.15. *If*

$$\sin b^n \pi x \to 0$$

as $n \to \infty$, *then*

$$x = \frac{p}{b^q}$$

where p *and* q *are integers; so that* $\sin b^n \pi x = 0$ *for* $n \geqq q$.

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* Hardy and Littlewood, *Contributions to the arithmetic theory of series*, Proceedings of the London Mathematical Society, ser. 2, vol. 11 (1912), pp. 411–478. The result required is obtained from Theorems 6 and 8 (*l. c.*, pp. 426–427) by supposing

$$\phi = \psi = y^{-\lambda}.$$

We have

$$b^n x = k_n + \epsilon_n ,$$

where k_n is an integer and $\epsilon_n \to 0$. It follows that

$$bk_n - k_{n+1} + b\epsilon_n - \epsilon_{n+1} = 0;$$

and it is evident that this is only possible if

$$k_{n+1} = bk_n , \qquad \epsilon_{n+1} = b\epsilon_n$$

from a certain value of n onwards, say for $n \geqq \nu$. We have therefore

$$\epsilon_{\nu+\mu} = b^{\mu} \epsilon_{\nu}$$

and so

$$b^{\mu} \epsilon_{\nu} \to 0$$

as $\mu \to \infty$. This is only possible if $\epsilon_{\nu} = 0$, which proves the lemma.

2.2. The proof of the final lemma is a little more elaborate. In stating it I use a notation introduced by Mr. Littlewood and myself:* I write

$$f = \Omega(\phi)$$

as signifying *the negation of* $f = o(\phi)$, that is to say as asserting the existence of a constant K such that

$$|f| > K\phi$$

for some special sequence of values whose limit is that to which the variable is supposed to tend.

LEMMA 2.21. *Suppose that*

$$f(y) = \sum b^{n\rho} e^{-b^n y} \sin b^n \pi x ,$$

where $y > 0$, *and that*

$$x \neq \frac{p}{b^q}$$

for any integral values of p *and* q. *Then*

$$f(y) = \Omega(y^{-\rho})$$

for all sufficiently large values of ρ.

I consider a special sequence of values of y, viz.,

(2.211) $$y = \frac{\rho}{b^m} \qquad (m = 1, 2, 3, \cdots);$$

and I write

(2.212) $$f(y) = \sum u_n = \sum_0^{m-1} u_n + u_m + \sum_{m+1}^{\infty} u_n = f_1 + f_2 + f_3 ,$$

* Hardy and Littlewood, *Some problems of diophantine approximation* (II), Acta Mathematica, vol. 37 (1914), pp. 193–238 (p. 225).

say. We have in the first place

$$
\begin{aligned}
|f_1| &< b^{m\rho}\, e^{-b^m y}\{b^{-\rho}\, e^{(b^m-b^{m-1})y} + b^{-2\rho}\, e^{(b^m-b^{m-2})y} + \cdots\} \\
&= b^{m\rho}\, e^{-\rho}\{e^{-\rho[\log b-1+(1/b)]} + e^{-\rho[2\log b-1+(1/b^2)]} + \cdots\} \\
&< b^{m\rho}\, e^{-\rho}\frac{e^{-B\rho}}{1-e^{-B\rho}},^{*}
\end{aligned}
\tag{2.2131}
$$

where

$$B = \log b - 1 + \frac{1}{b}. \tag{2.2132}$$

Similarly

$$
\begin{aligned}
|f_3| &< b^{m\rho}\, e^{-b^m y}\{b^{\rho}\, e^{-(b^{m+1}-b^m)y} + b^{2\rho}\, e^{-(b^{m+2}-b^m)y} + \cdots\} \\
&= b^{m\rho}\, e^{-\rho}\{e^{-\rho(b-1-\log b)} + e^{-\rho(b^2-1-2\log b)} + \cdots\} \\
&< b^{m\rho}\, e^{-\rho}\frac{e^{-B'\rho}}{1-e^{-B'\rho}},^{\dagger}
\end{aligned}
\tag{2.2141}
$$

where

$$B' = b - 1 - \log b. \tag{2.2142}$$

Now it follows from Lemma 2.15 that $\sin b^m \pi x$ does not tend to zero. There is therefore a positive constant c such that

$$|\sin b^m \pi x| > c \tag{2.215}$$

for an infinity of values of m, and we shall confine our attention to such values.

We can choose ρ_0 so that

$$\frac{e^{-B\rho}}{1-e^{-B\rho}} + \frac{e^{-B'\rho}}{1-e^{-B'\rho}} < \tfrac{1}{2}c \tag{2.216}$$

for $\rho > \rho_0$. Using (2.212), (2.2131), (2.2141), (2.216), and (2.215), we obtain:

$$
\begin{aligned}
|f| &\geqq |f_2| - |f_1| - |f_3| \\
&> b^{m\rho}\, e^{-\rho}(|\sin b^m \pi x| - \tfrac{1}{2}c) \\
&> Kb^{m\rho} > Ky^{-\rho},
\end{aligned}
$$

where the K's are constants, for $\rho > \rho_0$ and an infinity of values of y tending to zero. This completes the proof of the lemma.

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* Here we use the inequalities

$$B = \log b - 1 + \frac{1}{b} > 0$$

and

$$m \log b - 1 + \frac{1}{b^m} > m\left(\log b - 1 + \frac{1}{b}\right).$$

† Here we use the inequalities

$$B' = b - 1 - \log b > 0$$

and

$$b^m - 1 - m\log b > m(b - 1 - \log b).$$

2.3. We can now proceed to the proof of our main results. Let us suppose first that

$$ab > 1, \tag{2.311}$$

$$x \neq \frac{p}{b^q}, \tag{2.312}$$

and that

$$f(x) = \sum a^n \cos b^n \pi x = \sum a^n \cos b^n \theta = g(\theta) \tag{2.321}$$

satisfies the condition

$$f(x+h) - f(x) = o(|h|^{\xi}) \tag{2.3221}$$

or, what is the same thing

$$g(\theta+h) - g(\theta) = o(|h|^{\xi}), \tag{2.3222}$$

where

$$\xi = \frac{\log(1/a)}{\log b} < 1. \tag{2.323}$$

Then, if

$$G(r, \theta) = \sum a^n r^{b^n} \cos b^n \theta = \sum a^n e^{-b^n y} \cos b^n \pi x, \tag{2.33}$$

we have, by Lemma 2.11,

$$\begin{aligned} F(y) = \frac{\partial G}{\partial \theta} &= -\sum (ab)^n e^{-b^n y} \sin b^n \pi x \\ &= -\sum b^{(1-\xi)n} e^{-b^n y} \sin b^n \pi x \\ &= o(y^{\xi-1}) \end{aligned} \tag{2.34}$$

when $r \to 1$, $y \to 0$. We have also, by Lemma 2.14,

$$\begin{aligned} F^{(p)}(y) &= (-1)^{p+1} \sum (ab^{p+1})^n e^{-b^n y} \sin b^n \pi x \\ &= O \sum b^{(p+1-\xi)n} e^{-b^n y} \\ &= O(y^{\xi-p-1}), \end{aligned} \tag{2.35}$$

for all positive values of p. It follows, by Lemma 2.13, that

$$F^{(q)}(y) = o(y^{\xi-q-1}) \tag{2.36}$$

for $0 < q < p$, and therefore for all positive values of q. But this contradicts Lemma 2.21, if q is sufficiently large. The conditions (2.322) can therefore not be satisfied.

The case in which

$$ab = 1, \qquad \xi = 1, \tag{2.37}$$

may be treated in the same manner. The only difference is that we use Lemma 2.12 instead of Lemma 2.11, and that our final conclusion is that $f(x)$

cannot possess a finite differential coefficient for any value of x which is not of the form p/b^q.

2.4. This reasoning fails when $x = p/b^q$, and such values of x require special examination. We have in this case

$$\cos\{b^n\pi(x+h)\} = \cos(b^{n-q}p\pi + b^n\pi h) = \pm\cos b^n\pi h$$

for $n > q$, the negative sign being taken if b and p are both odd and the positive sign otherwise. The properties of the function in the neighborhood of such a value of x are therefore the same, for our present purpose, as those of the function

$$f(h) = \sum a^n \cos b^n\pi h \tag{2.41}$$

near $h = 0$. Now

$$\begin{aligned} f(h) - f(0) &= -2\sum a^n \sin^2 \tfrac{1}{2}b^n\pi h \\ &= -2(f_1 + f_2), \end{aligned}$$

where

$$f_1 = \sum_0^\nu a^n \sin^2 \tfrac{1}{2}b^n\pi h, \qquad f_2 = \sum_{\nu+1}^\infty a^n \sin^2 \tfrac{1}{2}b^n\pi h.$$

Choose ν so that

$$b^\nu|h| \leqq 1 < b^{\nu+1}|h|. \tag{2.42}$$

Then

$$f_1 + f_2 > f_1 > \sum_0^\nu a^n(b^n h)^2 = h^2\frac{(ab^2)^{\nu+1} - 1}{ab^2 - 1}$$

$$> Kh^2(ab^2)^\nu > Ka^\nu > Kb^{-\xi\nu} > K|h|^\xi,$$

where the K's are constants. It follows that

$$f(h) - f(0) \neq o(|h|^\xi).$$

If $ab > 1$, $\xi < 1$, we have proved that we want. In this case the graph of $f(h)$ has a cusp (pointing upwards) for $h = 0$, and that of Weierstrass's function has a cusp for $x = p/b^q$. If on the other hand $ab = 1$, $\xi = 1$, then we have proved that

$$\overline{\lim_{h\to+0}}\frac{f(h) - f(0)}{h} < 0, \qquad \underline{\lim_{h\to-0}}\frac{f(h) - f(0)}{h} > 0,$$

so that $f(h)$ has certainly no finite differential coefficient for $h = 0$, nor Weierstrass's function for $x = p/b^q$.

2.5. We have thus proved Theorems 1.31 and 1.33 in so far as they relate to the cosine-series and are of a negative character.

We have next to prove that, when $\xi < 1$, Weierstrass's function satisfies the condition

$$f(x+h) - f(x) = O(|h|^\xi)$$

for all values of x. We have

$$f(x+h)-f(x) = -2\sum a^n \sin\{b^n\pi(x+\tfrac{1}{2}h)\}\sin\tfrac{1}{2}b^n\pi h$$
$$= O\sum a^n|\sin\tfrac{1}{2}b^n\pi h|.$$

Choose ν as in (2.42). Then

$$f(x+h)-f(x) = O\left(|h|\sum_0^\nu a^n b^n + \sum_{\nu+1}^\infty a^n\right)$$
$$= O(a^\nu b^\nu|h| + a^\nu)$$
$$= O(a^\nu) = O(|h|^\xi).$$

The condition is therefore satisfied, and indeed uniformly in x. It should be observed that our argument fails when $ab=1$, $\xi=1$. In this case we can only assert that

$$f(x+h)-f(x) = O(\nu|h|+a^\nu) = O\left(|h|\log\frac{1}{|h|}\right).$$

It should also be observed that the argument of this paragraph applies to the sine series as well as to the cosine series, and is independent of the restriction that b is an integer.

2.6. The proof of Theorems 1.31 and 1.33 is now complete so far as the cosine series is concerned. The corresponding proof for the sine-series differs only in detail. The lemmas required are the same except that Lemma 2.15 must be replaced by

LEMMA 2.61. *If*

$$\cos b^n\pi x \to 0,$$

then b must be odd and

$$x = \frac{p+\frac{1}{2}}{b^q};$$

so that $\cos b^n\pi x = 0$ *from a certain value of n onwards:*

and that corresponding changes must be made in the wording of Lemma 2.21.

If now the value of x is not exceptional (i. e. one of those specified in Lemma 2.61), we can repeat the arguments of 2.3. It is therefore only necessary to discuss the exceptional values, which can exist only if b is odd. We have in this case

$$\sin\{b^n\pi(x+h)\} = \sin(b^{n-q}p\pi + \tfrac{1}{2}b^{n-q}\pi + b^n\pi h)$$
$$= \pm\sin(\tfrac{1}{2}b^{n-q}\pi + b^n\pi h),$$

for $n>q$, the sign being fixed as in 2.4. The last function is numerically equal to $\cos b^n\pi h$; it has always the same sign as $\cos b^n\pi h$, or always the opposite sign, if b is of the form $4k+1$; while if b is of the form $4k+3$ the signs agree and differ alternately. The problem is therefore reduced, either to the discussion of the function (2.41) near $h = 0$ (a discussion made already), or to that of

$$f(h) = \sum(-a)^n \cos b^n \pi h.$$

We have to prove that

$$f(h) - f(0) \neq o(|h|^\xi)$$

if $\xi < 1$, and that $f(h)$ has not a finite differential coefficient for $h = 0$, if $\xi = 1$.

I consider the special sequence of values

$$h = \frac{2}{b^\nu} \qquad (\nu = 1, 2, 3, \cdots).$$

We have

$$f(h) - f(0) = -2\sum_0^{\nu-1}(-a)^n \sin^2 \tfrac{1}{2} b^n \pi h$$

$$= (-1)^\nu 2a^{\nu-1} \sum_1^\nu \left(-\frac{1}{a}\right)^n \sin^2 \frac{\pi}{b^n}.$$

Now

$$\sum_1^\nu \left(-\frac{1}{a}\right)^n \sin^2 \frac{\pi}{b^n} \to \sum_1^\infty \left(-\frac{1}{a}\right)^n \sin^2 \frac{\pi}{b^n} = S,$$

say; and S being the sum of an alternating series of decreasing terms,* is positive.

Also

$$a^\nu = b^{-\xi\nu} = (\tfrac{1}{2}h)^\xi.$$

Thus $f(h) - f(0)$ is, for the particular sequence of values in question, greater in absolute value than a constant multiple of h^ξ, and alternately positive and negative. This completes the proof of Theorems 1.31 and 1.33.

* In order to prove this we must show that

$$\sin^2 \frac{\pi}{b^n} > \frac{1}{a} \sin^2 \frac{\pi}{b^{n+1}};$$

and this will certainly be true if

$$f(\theta) = \sin^2 \theta - b \sin^2 \frac{\theta}{b}$$

is positive for $0 < \theta \leqq \pi/b$. We have

$$f'(\theta) = \sin 2\theta - \sin \frac{2\theta}{b} = 2 \sin \frac{b-1}{b}\theta \cos \frac{b+1}{b}\theta,$$

which is positive for

$$0 < \theta < \frac{b\pi}{2(b+1)}.$$

Since

$$\frac{b}{2(b+1)} > \tfrac{1}{3}$$

if $b > 2$, this proves what we want, when $b \geqq 3$. When $b = 2$, we observe that $f(\theta)$ decreases from $\theta = \frac{1}{3}\pi$ to $\theta = \frac{1}{2}\pi$, and that

$$f(\tfrac{1}{2}\pi) = 1 - 2\left(\frac{1}{\sqrt{2}}\right)^2 = 0.$$

2.7. The possibility of the existence of a finite differential coefficient is thus disposed of in all cases. The question remains whether an equally comprehensive result holds for *infinite* differential coefficients. The theorem which follows, which includes theorem 1.32, shows that the answer to this question is negative.

THEOREM 2.71. *If*

$$ab \geqq 1, \qquad a(b+1) < 2$$

then the sine-series has the differential coefficient $+\infty$ *for* $x = 0$; *and if* b *is of the form* $4k+1$, *then the same is true of the cosine series for* $x = \frac{1}{2}$.

It is enough to prove the first of these statements, the second then following immediately by the transformation $x = \frac{1}{2} + y$.

We have

$$\frac{f(h) - f(0)}{h} = \frac{1}{h}\sum a^n \sin b^n \pi h \tag{2.711}$$

$$= \frac{1}{h}\sum_0^{\nu-1} a^n \sin b^n \pi h + \frac{1}{h}\sum_\nu^\infty a^n \sin b^n \pi h$$

$$= f_1 + f_2,$$

say, where ν is chosen so that

$$b^{\nu-1}|h| \leqq \tfrac{1}{2} < b^\nu |h|. \tag{2.712}$$

Suppose first that $ab > 1$. Then

$$f_1 > 2\sum_0^{\nu-1}(ab)^n = 2\frac{(ab)^\nu - 1}{ab - 1}, \tag{2.7131}$$

$$|f_2| < \frac{1}{|h|}\sum_\nu^\infty a^n = \frac{a^\nu}{(1-a)|h|}. \tag{2.7132}$$

Now

$$a(b+1) < 2, \qquad 1 - a > ab - 1;$$

so that

$$\frac{1-a}{ab-1} = 1 + \delta, \tag{2.714}$$

where $\delta > 0$. We can suppose h so small (or ν so large) that

$$\frac{(ab)^\nu - 1}{(ab)^\nu} > \frac{2+\delta}{2(1+\delta)}. \tag{2.715}$$

Then from (2.7131), (2.7132), (2.712), (2.714), and (2.715) it follows that

$$\frac{|f_2|}{f_1} < \frac{(ab)^\nu}{(ab)^\nu - 1}\,\frac{ab-1}{1-a} < \frac{1}{1+\frac{1}{2}\delta};$$

and so that $f_1 + f_2$ is greater than a constant multiple of f_1 or of $(ab)^\nu$. Thus

(2.716) $$\frac{f(h)-f(0)}{h} \to +\infty$$

as $h \to 0$.

If on the other hand $ab = 1$, then $|f_2|$ remains less than a constant and

$$f_1 > 2\nu \to +\infty,$$

so that (2.716) is still true.

When b is given, a number $\alpha(b)$ exists which is the *least* number such that the condition

$$ab > \alpha(b)$$

forbids the existence of a differential coefficient finite or infinite. All that we can say about $\alpha(b)$ at present is that

$$\frac{2}{b+1} \leqq \alpha(b) \leqq \frac{1+\frac{3}{2}\pi}{b+\frac{3}{2}\pi}.*$$

2.8. We have now proved everything that was stated in Section 1, subject to the restriction that b is an integer.

When b is not an integer, our series are no longer Fourier's series, and we can no longer employ Poisson's integral (2.111). The first stage in the discussion is naturally to construct a new formula to replace Poisson's. When we have done this, we find that some further modifications of the argument are needed, owing to the lack of any simple result corresponding to Lemmas 2.15 and 2.61, and the difficulty of determining precisely the exceptional values of x for which $\sin b^n \pi x \to 0$ or $\cos b^n \pi x \to 0$. It will be found, however, that no fundamental change in the method is necessary, and that the additional analysis required is not of an elaborate character.

3. Weierstrass's function when b is not an integer

3.1. I suppose now that b is any number greater than 1; and I write

(3.11) $$s = \sigma + it,$$

(as is usual in the theory of Dirichlet's series), and

(3.121) $$f(s) = \sum_1^\infty a^n e^{-b^n s} = G(\sigma, t) + iH(\sigma, t) \qquad (\sigma \geqq 0),$$

(3.122) $$G(0, t) = g(t).$$

3.2. Lemma 3.2. *If $\sigma > 0$ then*

$$G(\sigma, t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma g(u)}{\sigma^2 + (u-t)^2}\, du.$$

* The first inequality follows from 2.71, the second from (1.21).

We have, by a well-known formula,*

$$G(\sigma_0, t_0) = \frac{1}{2\pi}\int_C \left(\log \frac{r}{r'} \frac{dG}{dn} - G \frac{d}{dn} \log \frac{r}{r'} \right) dS, \tag{3.21}$$

where

$$G = G(\sigma, t),$$

$$r = \sqrt{(\sigma - \sigma_0)^2 + (t - t_0)^2},$$

$$r' = \sqrt{(\sigma + \sigma_0)^2 + (t - t_0)^2},$$

C is a closed contour which lies entirely in the half-plane $\sigma > 0$ and includes the point (σ_0, t_0) in its interior, and dS and dn are elements of the arc of C and the outward normal to C.

I take C to be the rectangle whose vertices are the points

$$(\delta, T), \qquad (\delta, -T), \qquad (\beta, -T), \qquad (\beta, T)$$

where

$$0 < \delta < \beta, \qquad T > 0,$$

and I denote the four sides of the rectangle, taken in order from the vertex (δ, T), by C_1, C_2, C_3, C_4, and the corresponding parts of the integral by J_1, J_2, J_3, J_4.

Suppose first that $\delta \to 0$. The functions G and

$$-\frac{\partial}{\partial\sigma} \log \frac{r}{r'} = -\frac{\sigma - \sigma_0}{(\sigma - \sigma_0)^2 + (t - t_0)^2} + \frac{\sigma + \sigma_0}{(\sigma + \sigma_0)^2 + (t - t_0)^2}$$

are continuous for $0 \leqq \sigma \leqq \delta$, $-T \leqq t \leqq T$. We have also, when $\sigma = \delta$,

$$\log \frac{r}{r'} = \log \sqrt{\frac{(\sigma_0 - \delta)^2 + (t - t_0)^2}{(\sigma_0 + \delta)^2 + (t - t_0)^2}} = O(\delta),$$

uniformly for $-T \leqq t \leqq T$; and

$$\frac{dG}{dn} = -\frac{\partial G}{\partial \sigma} = \sum (ab)^n e^{-b^n\sigma} \cos b^n t = O \sum (ab)^n e^{-b^n \sigma},$$

which is of the form

$$O(\sigma^{\xi - 1}) \qquad (0 < \xi < 1)$$

if $ab > 1$, and of the form

$$O\left(\log \frac{1}{\sigma}\right)$$

if $ab = 1$. From these facts it follows at once that

$$J_1 \to \mathcal{L}_1 = \frac{1}{\pi}\int_{-T}^{T} \frac{\sigma_0 \, g(t)}{\sigma_0^2 + (t - t_0)^2} dt,$$

* See for example Picard, *Traité d'analyse*, vol. 2, pp. 15, 16.

and that J_2 and J_4 tend to limits $\mathcal{L}_2$ and $\mathcal{L}_4$, the former of these limits, for example, being given by the formula

$$\mathcal{L}_2 = \frac{1}{2\pi}\int_0^\beta \left(\log\frac{r}{r'}\frac{dG}{dn} - G\frac{d}{dn}\log\frac{r}{r'}\right)d\sigma,$$

wherein $t = -T$.

Suppose next that $T \to \infty$. Then we have, in $\mathcal{L}_3$,

$$G = O(1), \qquad \frac{dG}{dn} = -\frac{\partial G}{\partial t} = \frac{O(1)^*}{\sigma^{1-\xi}},$$

$$\log\frac{r}{r'} = O\left(\frac{1}{T^2}\right), \qquad \frac{d}{dn}\log\frac{r}{r'} = -\frac{\partial}{\partial t}\log\frac{r}{r'} = O\left(\frac{1}{T^3}\right),$$

uniformly for $0 \leqq \sigma \leqq \beta$. Thus $\mathcal{L}_2 \to 0$, and similarly $\mathcal{L}_4 \to 0$. Also $\mathcal{L}_1$ and J_3 obviously tend to limits $\mathcal{L}$ and $\mathcal{H}$. The formula (3.21) thus passes over into

(3.22) $$G(\sigma_0, t_0) = \mathcal{L} + \mathcal{H},$$

where

(3.23) $$\mathcal{L} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_0\, g(t)}{\sigma_0^2 + (t-t_0)^2}dt,$$

(3.24) $$\mathcal{H} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\log\frac{r}{r'}\frac{dG}{dn} - G\frac{d}{dn}\log\frac{r}{r'}\right)dt,$$

the value of σ in $\mathcal{H}$ being β.

Finally we make $\beta \to \infty$. If we observe that

$$\log\frac{r}{r'} = O\left(\frac{1}{T^2}\right), \qquad \frac{d}{dn}\log\frac{r}{r'} = \frac{\partial}{\partial\sigma}\log\frac{r}{r'} = O\left(\frac{1}{T^2}\right),$$

uniformly for $\sigma \geqq \beta$; and that

$$G = O(e^{-b\sigma}), \qquad \frac{dG}{dn} = \frac{\partial G}{\partial\sigma} = O(e^{-b\sigma}),$$

uniformly in T; we see that $\mathcal{H} \to 0$. Thus the proof of the lemma is completed.

3.3. Let us suppose first that $ab > 1$. In this case we shall use, instead of Lemma 2.11, the two lemmas which follow.

LEMMA 3.31. *If*

$$g(t) - g(t_0) = o(|t-t_0|^\alpha),$$

where $0 < \alpha < 1$, *when* $t \to t_0$, *then*

* This must be replaced by

$$O(1)\log\frac{1}{\sigma}$$

when $ab = 1$.

$$\frac{\partial G(\sigma, t_0)}{\partial t_0} = o(\sigma^{\alpha-1}),$$

when $\sigma \to 0$.

LEMMA 3.32. *Under the same conditions*

$$\frac{\partial G(\sigma, t_0)}{\partial \sigma} = o(\sigma^{\alpha-1}).$$

The proofs of these lemmas are very similar, and the first is in all essentials the same as that of Lemma 2.11. It will therefore be sufficient to give the proof of the second.

We may take $t_0 = 0$.* We have then

$$\frac{\partial G}{\partial \sigma} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t^2 - \sigma^2}{(t^2 + \sigma^2)^2} g(t)\, dt. \tag{3.321}$$

If we write

$$g(t) - g(0) = \gamma(t),$$

and observe that

$$\int_{-\infty}^{\infty} \frac{t^2 - \sigma^2}{(t^2 + \sigma^2)^2} dt = 0,$$

we see that

$$\frac{\partial G}{\partial \sigma} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t^2 - \sigma^2}{(t^2 + \sigma^2)^2} \gamma(t)\, dt. \tag{3.322}$$

Choose δ so that

$$|\gamma(t)| < \epsilon |t|^\alpha$$

for $-\delta \leqq t \leqq \delta$. Then, if we write

$$\frac{1}{\pi}\int_{-\infty}^{\infty} = \frac{1}{\pi}\left(\int_{-\infty}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty}\right) = \frac{1}{\pi}(J_1 + J_2 + J_3), \tag{3.323}$$

we have

$$J_1 = O(1), \qquad J_3 = O(1), \tag{3.324}$$

and

$$\begin{aligned} |J_2| &< \epsilon \int_{-\delta}^{\delta} \frac{|t^2 - \sigma^2|}{(t^2 + \sigma^2)^2} |t|^\alpha\, dt \\ &< \epsilon \int_{-\infty}^{\infty} \frac{|t^2 - \sigma^2|}{(t^2 + \sigma^2)^2} |t|^\alpha\, dt < K\epsilon\sigma^{\alpha-1}, \end{aligned} \tag{3.325}$$

where K is a constant. The truth of the lemma follows immediately from (3.322)–(3.325).

---

* The point $t = 0$ has of course a special character for the particular function $g(t)$ which we are considering. This is in no way relevant to the proof of the lemma, which is, like Lemma 2.11, a proposition in the general theory of functions, the truth of which depends only on the validity of the fundamental integral formula, the facts that $g(t)$ is continuous and bounded, and the special hypothesis of the lemma itself. There is therefore no loss of generality in supposing that $t_0 = 0$.

3.4. Suppose now that

$$g(t+h) - g(t) = o(|h|^{\xi}).$$

Then, by Lemmas 3.31 and 3.32, we have

$$\frac{\partial G}{\partial t} = -\sum (ab)^n e^{-b^n\sigma} \sin b^n t = o(\sigma^{\xi-1})$$

and

$$\frac{\partial G}{\partial \sigma} = -\sum (ab)^n e^{-b^n\sigma} \cos b^n t = o(\sigma^{\xi-1}),$$

and so

$$f(y) = \sum (ab)^n e^{-b^n(\sigma+it)} = o(\sigma^{\xi-1}).$$

We can now obtain a contradiction by following the argument of 2.3. It is only necessary to observe that Lemma 2.13 holds for complex as well as for real functions of a real variable, and to use, instead of Lemma 2.21, the proposition: *if*

$$f(y) = \sum b^{n\rho} e^{-b^n(\sigma+it)} \qquad (\sigma > 0)$$

then

$$f(y) = \Omega(\sigma^{-\rho})$$

for all sufficiently large values of ρ. Since $|e^{-b^n it}| = 1$, there is no longer any question of exceptional values of t.

3.5. Now suppose that $ab = 1$. Instead of Lemma 3.31 we use the following lemma, which corresponds to Lemma 2.12.

LEMMA 3.51. *If* $g(t)$ *possesses a finite differential coefficient* $g'(t_0)$ *for* $t = t_0$, *then*

$$\frac{\partial G(\sigma, t_0)}{\partial t_0} \to g'(t_0)$$

when $\sigma \to 0$.

The proof of this presents no fresh difficulty. But it is not necessarily true that

$$\frac{\partial G(\sigma, t_0)}{\partial \sigma}$$

tends to a limit.* It is therefore necessary to follow a line of argument which differs slightly from that of 3.4.

* The difference arises from the fact that the integral

$$\int^{\infty} \frac{2\sigma t}{(\sigma^2 + t^2)^2} t dt$$

is convergent, but the integral

$$\int^{\infty} \frac{\sigma^2 - t^2}{(\sigma^2 + t^2)^2} t dt$$

divergent. All that we can prove is that

$$\frac{\partial G}{\partial \sigma} = o\left(\log \frac{1}{\sigma}\right),$$

and this is not sufficient for our argument.

LEMMA 3.52. *Under the same conditions as those of Lemma 3.51, we have*

$$\frac{\partial^2 G(\sigma, t_0)}{\partial t_0^2} = o\left(\frac{1}{\sigma}\right).$$

Taking $t_0 = 0$, we find

$$\frac{\partial^2 G}{\partial t_0^2} = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sigma(3t^2 - \sigma^2)}{(\sigma^2 + t^2)^3} g(t)\, dt.$$

In this equation we can replace $g(t)$ by $\gamma(t)$, since

$$\int_{-\infty}^{\infty} \frac{\sigma(3t^2 - \sigma^2)}{(\sigma^2 + t^2)^3}\, dt = 0;$$

and the proof of the lemma may then be completed by arguments similar to those used in the proof of Lemma 3.32.*

3.6. Suppose now that $g(t)$ possesses a finite differential coefficient $g'(t)$, and write

$$f(\sigma) = \frac{\partial G}{\partial t} = -\sum e^{-b^n\sigma} \sin b^n t.$$

Then, by Lemma 3.51, we have

$$f(\sigma) = g'(t) + o(1)$$

when $\sigma \to 0$. But we have also, by Lemma 2.14:

$$f''(\sigma) = -\sum b^{2n} e^{-b^n\sigma} \sin b^n t = O\left(\frac{1}{\sigma^2}\right),$$

and therefore, by Lemma 2.13,

$$f'(\sigma) = \sum b^n e^{-b^n\sigma} \sin b^n t = o\left(\frac{1}{\sigma}\right). \tag{3.61}$$

On the other hand, by Lemma 3.52, we have

$$\frac{\partial^2 G}{\partial t^2} = -\sum b^n e^{-b^n\sigma} \cos b^n t = o\left(\frac{1}{\sigma}\right). \tag{3.62}$$

From (3.61) and (3.62) it follows that

$$F(\sigma) = \sum b^n e^{-b^n(\sigma + it)} = o\left(\frac{1}{\sigma}\right). \tag{3.63}$$

Also, by Lemma 2.14, we have

$$F^{(p)}(\sigma) = (-1)^p \sum b^{(p+1)n} e^{-b^n(\sigma + it)} = O\left(\frac{1}{\sigma^{p+1}}\right) \tag{3.64}$$

* The critical part of the integral is less than a constant multiple of

$$\epsilon \int_{-\delta}^{\delta} \frac{\sigma\,|\,\sigma^2 - 3t^2\,|}{(\sigma^2 + t^2)^3}\,|\,t\,|\, dt,$$

or than a constant multiple of ϵ/σ.

for all values of p. Hence, as in 2.3, it follows that the O may be replaced by o; and this, as before, leads to a contradiction.

3.7. It is only necessary to add the following remarks. Our argument, throughout this section, has been stated in terms of Weierstrass's cosine series. The same arguments apply to the sine series, as there are now no "exceptional values," and it was only the existence of such values which differentiated the two cases in Section 2. The positive statement in Theorem 1.33 has already been proved, the proof given in 2.5 applying to all values of b. No fresh proof is required of Theorem 1.32. The proofs of the theorems stated in 1.3 are therefore now complete in all cases.

4. Other functions

4.1. **A function which does not satisfy a Lipschitz condition of any order.** It was suggested to me recently by Dr. Marcel Riesz that it would be of interest to have an example of an absolutely convergent Fourier's series whose sum does not satisfy any condition of the type

$$f(x+h)-f(x) = O(|h|^{\alpha}) \qquad (\alpha > 0)$$

for any value of x. The function

$$f(x) = \sum \frac{\cos b^n \pi x}{n^2}$$

is such a function. It is in fact easy to prove, by the methods used in this paper, that

$$f(x+h)-f(x) \neq o\left(\frac{1}{|\log|h||}\right)^2.$$

I should observe, however, that a somewhat less simple example may be found by merely combining remarks made by Faber and Landsberg in their papers quoted on p. 301. Faber writes

$$F(x) = \sum 10^{-n}\phi(2^{n!x}), \tag{4.11}$$

where $\phi(x)$ is the function of period 1 which is equal to x for $0 \leqq x \leqq \frac{1}{2}$ and to $1-x$ for $\frac{1}{2} \leqq x \leqq 1$, and he shows that

$$F(x+h)-F(x) \neq O\left(\frac{1}{|\log|h||}\right).$$

Landsberg, on the other hand, uses the expansion of a function, substantially equivalent to $\phi(x)$, in a Fourier's series. We have in fact

$$\phi(x) = \frac{1}{4} - \frac{2}{\pi^2}\sum\frac{\cos 2\nu\pi x}{\nu^2} \qquad (\nu = 1, 3, 5 \cdots).$$

If we substitute this expansion in (4.11), we obtain an expansion of $F(x)$ as an absolutely convergent Fourier series, and so an example of the kind required.

4.2. **On a theorem of S. Bernstein.** In this connection it is natural to allude to an important theorem of S. Bernstein, which may be proved very simply by the use of the ideas of this paper. Bernstein's* theorem is as follows:

If $f(x)$ satisfies a Lipschitz condition of order α, where $\alpha > \frac{1}{2}$, throughout the interval $(0, 1)$, i. e. if

$$|f(x+h) - f(x)| < K|h|^{\alpha},$$

where K is an absolute constant, then the Fourier series of $f(x)$ is absolutely convergent. Also $\frac{1}{2}$ is the least number which possesses this property.

Suppose that $2\pi x = \theta$ and

$$f(x) = g(\theta) = \tfrac{1}{2}a_0 + \sum (a_n \cos n\theta + b_n \sin n\theta);$$

and let

$$G(r, \theta) = \tfrac{1}{2}a_0 + \sum r^n (a_n \cos n\theta + b_n \sin n\theta),$$

if $r < 1$, and $G(1, \theta) = g(\theta)$. Then $G(r, \theta)$ is continuous for $0 \leqq r \leqq 1$, $0 \leqq \theta \leqq 2\pi$.

It follows from a simple modification of Lemma 2.11† that

$$\frac{\partial G}{\partial \theta} = -\sum n r^{n-1} (a_n \sin n\theta - b_n \cos n\theta) = O\{(1-r)^{\alpha-1}\},$$

uniformly in θ. Squaring, and integrating from $\theta = 0$ to $\theta = 2\pi$, we obtain

$$\sum n^2 r^{2n} (|a_n|^2 + |b_n|^2) = O(1-r)^{2\alpha-2}.$$

Hence, by putting $r = 1 - (1/\nu)$, we obtain

$$\sum_1^{\nu} n^2 (|a_n|^2 + |b_n|^2) = O(\nu^{2-2\alpha}),$$

and so, by Schwarz's inequality,

$$\sum_1^{\nu} n (|a_n| + |b_n|) = O(\nu^{\frac{3}{2}-\alpha}). \tag{4.21}$$

From (4.21) it is easy to deduce that the series

$$\sum_1^{\nu} n^{\beta} (|a_n| + |b_n|)$$

is convergent if $\beta < \alpha - \frac{1}{2}$.

* *Sur la convergence absolue des séries trigonométriques*, Comptes Rendus, June 8, 1914.

† With O in the place of o, and "uniformly" inserted in premises and conclusion.

This establishes the truth of the first part of Bernstein's Theorem (indeed rather more). The truth of the second part may be shown by the example of the function

$$g(\theta) = \sum n^{-b} \cos(n^a + n\theta),$$

where $0 < a < 1$, $0 < b < 1$. In this case $G(r, \theta)$ is the real part of

$$F(z) = F(re^{i\theta}) = \sum n^{-b} e^{in^a} z^n.$$

I have shown elsewhere* that this function is continuous for

$$|z| \leqq 1$$

if

$$\tfrac{1}{2}a + b > 1:$$

and it is not difficult to go further and to show that $g(\theta)$ satisfies a Lipschitz condition of order $\frac{1}{2}a + b - 1$.† Now let α be any number less than $\frac{1}{2}$. Then we can choose numbers a and b, each less than 1, and such that

$$\tfrac{1}{2}a + b - 1 > \alpha.$$

The function $g(\theta)$ then satisfies a Lipschitz condition of order greater than α, but its Fourier series is *not* absolutely convergent.

4.3. **Riemann's non-differentiable function.** It was supposed by Riemann‡

* Hardy, *A theorem concerning Taylor's series*, Quarterly Journal of Mathematics, vol. 44 (1913), pp. 147–160. I take this opportunity of correcting a misprint: on p. 153, lines 1 and 6, *for* $(-z)^{(1-b)/a}$ *read* $(-z)^{-(1-b)/a}$.

† The example shows the importance of the distinction between a Lipschitz condition satisfied *at every point of an interval* and one satisfied (uniformly) *throughout the interval.* Since $f(z)$ is regular save for $z = 1$, $g(\theta)$ satisfies a condition of order 1 for every value of θ that is not a multiple of 2π. And it is easy to prove that, at the point $\theta = 0$, it satisfies a condition of order

$$\beta = \text{Min}\left(1, \frac{\frac{1}{2}a + b - 1}{1 - a}\right).$$

Suppose, *e. g.*, that

$$a = \tfrac{3}{4}, \qquad b = \tfrac{7}{8}, \qquad \tfrac{1}{2}a + b - 1 = \tfrac{1}{4}, \qquad \frac{\frac{1}{2}a + b - 1}{1 - a} = 1.$$

Then the function satisfies a condition of order 1 at every point. But it does not satisfy *throughout the interval* $(0, 2\pi)$ any condition of order greater than $\frac{1}{4}$.

This remark was suggested to me by an observation of Dr. Marcel Riesz, viz. that the functions derived from the expansion of

$$(1 - z)^{\lambda} e^{-1/(1-z)}$$

possess similar peculiarities.

‡ My authority for this statement is du Bois-Reymond, who, in his memoir quoted in Section 1, remarks (*l. c.*, p. 28): " Ist seit einigen Jahren wohl hauptsächlich in Deutschlands mathematischen Kreisen von der Möglichkeit von Functionen ohne Differential-quotienten die Rede, besonders seitdem *Riemannsche* Schüler verkündeten, ihr Lehrer habe von der Reihe mit dem gliede $(\sin p^2 x)/p^2$ die Nichtdifferentiirbarkeit behauptet. Diese Reihe solle für gewisse, in jedem noch so kleinen Intervalle unbegrenzt oft wiederkehrende Werthe von x keinen endlichen bestimmten Differentialquotienten zulassen. Einen Beweis hierfür hat unseres Wissens keiner der *Riemannschen* Schüler zu Papiere gebracht, indessen ist nach

that the function

$$f(x) = \sum \frac{\sin n^2 \pi x}{n^2}$$

has no finite differential coefficient for any one of an everywhere dense set of values of x. No proof or disproof of this assertion has, so far as I know, been published. The question is a much more difficult one than any of those connected with Weierstrass's function, owing to the comparatively slow increase of the sequence n^2. But a combination of the methods used earlier in this paper with certain results proved elsewhere* by Mr. Littlewood and myself has led me to a proof of Riemann's assertion and a good deal more.

Suppose that Riemann's function is differentiable† for a certain value of x. Then, by Lemma 2.12, we have

$$\sum r^{n^2} \cos n^2 \pi x = A + o(1),$$

where A is a constant, as $r \to 1$.

But

$$\sum r^{n^2} \cos n^2 \pi x = \Omega\{(1-r)^{-\frac{1}{4}}\}$$

if x is irrational,‡ and

$$\sum r^{n^2} \cos n^2 \pi x = \Omega\{(1-r)^{-\frac{1}{2}}\}$$

if x is a rational of the form $(2\lambda + 1)/2\mu$ or $2\lambda/(4\mu + 1)$.§ Thus Riemann's function is certainly not differentiable for any irrational (and some rational) values of x. It is easy, by using Lemma 2.11, instead of Lemma 2.12, to show that Riemann's function cannot satisfy the condition

$$f(x+h) - f(x) = o(|h|^{\frac{3}{4}})$$

for any irrational x.

We can prove more, viz.,

THEOREM 4.31. *Neither of the functions*

$$\sum \frac{\cos n^2 \pi x}{n^a}, \qquad \sum \frac{\sin n^2 \pi x}{n^a},$$

where $\alpha < \frac{5}{2}$, is differentiable for any irrational value of x.

einer Mitteilung des Herrn *Weierstrass* die *Riemannsche* Behauptung richtig." I am not clear as to the meaning of the last remark. It may mean that Weierstrass, in some communication now lost, had investigated Riemann's function itself. Or it may mean merely that Weierstrass's example showed that Riemann was right in his general view as to the existence of continuous non-differentiable functions.

* Hardy and Littlewood, *Some problems of diophantine approximation* (II), Acta Mathematica, vol. 37 (1914), pp. 193–238.

† In what follows I use "differentiable" as implying the existence of a finite differential coefficient.

‡ *L. c.*, p. 233.

§ *Ibid.*, p. 195. This last result is trivial: that concerning irrational values lies much deeper.

Suppose, e. g., that the sine-series is differentiable. Then, by Lemma 2.12, we have

$$\sum n^{2-\alpha} r^{n^2} \cos n^2 \pi x = A + o(1),$$

or

$$f(y) = \sum n^{2-\alpha} e^{-n^2 y} \cos n^2 \pi x = A + o(1). \tag{4.311}$$

But

$$f^{(p)}(y) = (-1)^p \sum n^{2p+2-\alpha} e^{-n^2 y} \cos n^2 \pi x$$
$$= O \sum n^{2p+2-\alpha} e^{-n^2 y} = O(y^{-p-\frac{3}{2}+\frac{1}{2}\alpha}).$$

Hence, by the theorem of Mr. Littlewood and myself quoted on p. 323,* we have

$$f^{(q)}(y) = o\{y^{-\frac{q}{p}(p+\frac{3}{2}-\frac{1}{2}\alpha)}\},$$

for $0 < q < p$, and in particular

$$f'(y) = o(y^{-1-\frac{3}{2p}+\frac{\alpha}{2p}}). \tag{4.312}$$

But it is easy to prove that

$$f'(y) = -\sum n^{4-\alpha} e^{-n^2 y} \cos n^2 \pi x = \Omega(y^{-\frac{9}{4}+\frac{\alpha}{2}}).\dagger \tag{4.313}$$

From (4.312) and (4.313) it follows that

$$1 + \frac{3}{2p} - \frac{\alpha}{2p} > \frac{9}{4} - \frac{\alpha}{2},$$

and this is impossible if $\alpha < (\frac{5}{2})$ and p is sufficiently large.

It follows from Theorem 4.31 that *the series*

$$\sum \frac{\cos n^2 \pi x}{n^\beta}, \qquad \sum \frac{\sin n^2 \pi x}{n^\beta},$$

where $0 < \beta < \frac{1}{2}$, *are not Fourier's series.* For if the first (e. g.) were a Fourier's series, then the sum of the integrated series

$$\sum \frac{\sin n^2 \pi x}{n^{2+\beta}}$$

would be a function of limited total fluctuation, and would therefore be

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* Of which Lemma 2.13 is a special case.

† The contrary hypothesis would involve

$$\Sigma e^{-n^2 y} \cos n^2 \pi x = o(y^{-\lambda}),$$

where

$$\lambda = \frac{9}{4} - \frac{\alpha}{2} - \frac{4-\alpha}{2} = \frac{1}{4};$$

and this is false.

differentiable almost everywhere. This result was proved by Mr. Littlewood and myself, in a different manner, in our paper referred to above.*

I may observe in conclusion that it is easy to prove directly that the function

$$f(x) = \sum \frac{\sin n^2 \pi x}{n^a},$$

where $2 < \alpha < (\frac{5}{2})$ has the differential coefficient $+\infty$ for $x = 0$. A similar direct method could no doubt be applied to an everywhere dense set of rational values of x.

* *L. c.*, p. 237.