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Counterexamples in Analysis

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Chapter 3
Differentiation

Introduction

In some of the examples of this chapter the word *derivative* is permitted to be applied to the infinite limits

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = +\infty, \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = -\infty.$$

However, the term *differentiable function* is used only in the strict sense of a function having a finite derivative at each point of its domain. A function is said to be **infinitely differentiable** iff it has (finite) derivatives of all orders at every point of its domain.

The exponential function with base e is alternatively denoted e^x and exp (x).

As in Chapter 2, all sets, including domains and ranges, will be assumed to be subsets of $\mathbb{R}$ unless explicit statement to the contrary is made. This assumption will remain valid through Part I of this book, that is, through Chapter 8.

1. A function that is not a derivative.

The signum function (cf. the Introduction, Chapter 1) or, indeed, any function with jump discontinuities, has no primitive — that is, fails to be the derivative of any function — since it fails to have the intermediate value property enjoyed by continuous functions and derivatives alike (cf. [34], p. 84, Ex. 40). An example of a discontinuous derivative is given next.

signum function

$\mathrm{sgn}(\cdot) : \mathbb{R} \to \mathbb{R}$

$$\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

2. A differentiable function with a discontinuous derivative.
The function

$$f(x) \equiv \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

has as its derivative the function

$$f'(x) = \begin{cases} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

which is discontinuous at the origin.

3. A discontinuous function having everywhere a derivative (not necessarily finite).
For such an example to exist the definition of derivative must be extended to include the limits $\pm\infty$. If this is done, the discontinuous signum function (Example 1) has the derivative

$$g(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ +\infty & \text{if } x = 0. \end{cases}$$

4. A differentiable function having an extreme value at a point where the derivative does not make a simple change in sign.
The function

$$f(x) \equiv \begin{cases} x^4 \left(2 + \sin \dfrac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

has an absolute minimum value at $x = 0$. Its derivative is

$$f'(x) \equiv \begin{cases} x^2 \left[4x \left(2 + \sin \dfrac{1}{x}\right) - \cos \dfrac{1}{x}\right] & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

which has both positive and negative values in every neighborhood of the origin. In no interval of the form $(a, 0)$ or $(0, b)$ is f monotonic.

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5. A differentiable function whose derivative is positive at a point but which is not monotonic in any neighborhood of the point.

The function

$$f(x) \equiv \begin{cases} x + 2x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

has the derivative

$$f'(x) = \begin{cases} 1 + 4x \sin \dfrac{1}{x} - 2 \cos \dfrac{1}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

In every neighborhood of 0 the function $f'(x)$ has both positive and negative values.

6. A function whose derivative is finite but unbounded on a closed interval.

The function

$$f(x) \equiv \begin{cases} x^2 \sin \dfrac{1}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

has the derivative

$$f'(x) = \begin{cases} 2x \sin \dfrac{1}{x^2} - \dfrac{2}{x} \cos \dfrac{1}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

which is unbounded on $[-1, 1]$.

7. A function whose derivative exists and is bounded but possesses no (absolute) extreme values on a closed interval.

The function

$$f(x) \equiv \begin{cases} x^4 e^{-\frac{1}{4}x^2} \sin \dfrac{8}{x^3} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

has the derivative

$$f'(x) = \begin{cases} e^{-\frac{1}{4}x^2}\left[(4x^3 - \frac{1}{2}x^5)\sin\dfrac{8}{x^3} - 24\cos\dfrac{8}{x^3}\right] & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

In every neighborhood of the origin this derivative has values arbitrarily near both 24 and -24. On the other hand, for $0 < h \equiv |x| \leqq 1$ (cf. [34], p. 83, Ex. 29),

$$0 < e^{-\frac{1}{4}x^2} < 1 - \frac{1}{4}h^2 e^{-\frac{1}{4}h^2} < 1 - \frac{3}{16}h^2,$$

and

$$\left|\left(4x^3 - \frac{1}{2}x^5\right)\sin\frac{8}{x^3} - 24\cos\frac{8}{x^3}\right| \leqq 24 + \frac{9}{2}h^3.$$

Therefore $0 < h \leqq 1$ implies

$$|f'(x)| < \left(1 - \frac{3}{16}h^2\right)\left(24 + \frac{9}{2}h^3\right) < 24 - \frac{9}{2}h^2(1 - h) \leqq 24.$$

Therefore, on the closed interval $[-1, 1]$ the range of the function f' has supremum equal to 24 and infimum equal to -24, and neither of these numbers is assumed as a value of f'.

8. A function that is everywhere continuous and nowhere differentiable.

The function $|x|$ is everywhere continuous but it is not differentiable at $x = 0$. By means of translates of this function it is possible to define everywhere continuous functions that fail to be differentiable at each point of an arbitrarily given finite set. In the following paragraph we shall discuss an example using an infinite set of translates of the function $|x|$.

The function of Example 21, Chapter 2, is nowhere differentiable. To see this let a be an arbitrary real number, and for any positive integer n, choose h_n to be either 4^{-n-1} or -4^{-n-1} so that $|f_n(a + h_n) - f_n(a)| = |h_n|$. Then $|f_m(a + h_n) - f_m(a)|$ has this same value $|h_n|$ for all $m \leqq n$, and vanishes for $m > n$. Hence the difference quotient $(f(a + h_n) - f(a))/h_n$ is an integer that is even if n is even

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and odd if n is odd. It follows that

$$\lim_{n\to+\infty} \frac{f(a+h_n) - f(a)}{h_n}$$

cannot exist, and therefore that $f'(a)$ cannot exist as a finite limit.

The first example of a continuous nondifferentiable function was given by K. W. T. Weierstrass (German, 1815–1897):

$$f(x) = \sum_{n=0}^{+\infty} b^n \cos(a^n \pi x),$$

where b is an odd integer and a is such that $0 < a < 1$ and $ab > 1 + \frac{3}{2}\pi$. The example presented above is a modification of one given in 1930 by B. L. Van der Waerden (cf. [48], p. 353). There are now known to be examples of continuous functions that have nowhere a one-sided finite or infinite derivative. For further discussion of these examples, and references, see [48], pp. 350–354, [10], pp. 61–62, 115, 126, and [21], vol. II, pp. 401–412.

The present example, as described in Example 21, Chapter 2, was shown to be nowhere monotonic. For an example of a function that is everywhere differentiable and nowhere monotonic, see [21], vol. II, pp. 412–421. Indeed, this last example gives a very elaborate construction of a function that is everywhere differentiable and has a dense set of relative maxima and a dense set of relative minima.

9. A differentiable function for which the law of the mean fails.

Again, we must go beyond the real number system for the range of such a function. The complex-valued function of a real variable x, defined

$$f(x) \equiv \cos x + i \sin x,$$

is everywhere continuous and differentiable (cf. [34], pp. 509–513), but there is no interval $[a, b]$, where $a < b$, such that for some ξ between a and b,

$$(\cos b + i \sin b) - (\cos a + i \sin a) = (-\sin \xi + i \cos \xi)(b - a).$$

Assuming that the preceding equation *is* possible, we equate the squares of the moduli (absolute values) of the two members:

$$(\cos b - \cos a)^2 + (\sin b - \sin a)^2 = (b - a)^2$$

or, with the aid of elementary identities:

$$\sin^2 \frac{b-a}{2} = \left(\frac{b-a}{2}\right)^2.$$

Since there is no positive number h such that $\sin h = h$ (cf. [34], p. 78), a contradiction has been obtained.

10. An infinitely differentiable function of x that is positive for positive x and vanishes for negative x.

The function

$$f(x) \equiv \begin{cases} e^{-1/x^2} & \text{if } x > 0, \\ 0 & \text{if } x \leqq 0 \end{cases}$$

is infinitely differentiable, all of its derivatives at $x = 0$ being equal to 0 (cf. [34], p. 108, Ex. 52).

11. An infinitely differentiable function that is positive in the unit interval and vanishes outside.

$$f(x) \equiv \begin{cases} e^{-1/x^2(1-x)^2} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

12. An infinitely differentiable "bridging function," equal to 1 on $[1, +\infty)$, equal to 0 on $(-\infty, 0]$, and strictly monotonic on $[0, 1]$.

$$f(x) = \begin{cases} \exp\left[-\frac{1}{x^2}\exp\left(-\frac{1}{(1-x)^2}\right)\right] & \text{if } 0 < x < 1, \\ 0 & \text{if } x \leqq 0, \\ 1 & \text{if } x \geqq 1. \end{cases}$$

13. An infinitely differentiable monotonic function f such that

$$\lim_{x \to +\infty} f(x) = 0, \qquad \lim_{x \to +\infty} f'(x) \neq 0.$$

If the word *monotonic* is deleted there are trivial examples, for instance $(\sin x^2)/x$. For a monotonic example, let $f(x)$ be defined to

be equal to 1 for $x \leqq 1$, equal to $1/n$ on the closed interval $[2n - 1, 2n]$, for $n = 1, 2, \cdots$, and on the intervening intervals $(2n, 2n + 1)$ define $f(x)$ by translations of the bridging function of Example 12, with appropriate negative factors for changes in the vertical scale.

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for Example 8

21. A continuous function that is nowhere monotonic.

Let $f_1(x) \equiv |x|$ for $|x| \leqq \frac{1}{2}$, and let $f_1(x)$ be defined for other values of x by periodic continuation with period 1, i.e., $f_1(x+n) = f_1(x)$ for every real number x and integer n. For $n > 1$ define $f_n(x) \equiv 4^{-n+1}f_1(4^{n-1}x)$, so that for every positive integer n, f_n is a periodic function of period 4^{-n+1}, and maximum value $\frac{1}{2}\cdot 4^{-n+1}$. Finally, define f with domain $\mathfrak{R}$:

$$f(x) \equiv \sum_{n=1}^{+\infty} f_n(x) = \sum_{n=1}^{+\infty} \frac{f_1(4^{n-1}x)}{4^{n-1}}.$$

Since $|f_n(x)| \leqq \frac{1}{2}\cdot 4^{-n+1}$, by the Weierstrass M-test this series converges uniformly on $\mathfrak{R}$, and f is everywhere continuous. For any point a of the form $a = k \cdot 4^{-m}$, where k is an integer and m is a positive integer, $f_n(a) = 0$ for $n > m$, and hence $f(a) = f_1(a) + \cdots + f_m(a)$. For any positive integer m, let h_m be the positive number 4^{-2m-1}. Then $f_n(a + h_m) = 0$ for $n > 2m+1$, and hence

$$\begin{aligned} f(a+h_m) - f(a) &= [f_1(a+h_m) - f_1(a)] + \cdots \\ &\qquad + [f_m(a+h_m) - f_m(a)] \\ &\quad + f_{m+1}(a+h_m) + \cdots + f_{2m+1}(a+h_m) \\ &\geqq -mh_m + (m+1)h_m = h_m > 0. \end{aligned}$$

Similarly,

$$f(a-h_m) - f(a) \geqq -mh_m + (m+1)h_m = h_m > 0.$$

Since members of the form $a = k\cdot 4^{-m}$ are dense, it follows that in no open interval is f monotonic.

The above typifies constructions involving the condensation of singularities.