Bounded Sequences

Def. A sequence $\{s_n\}_n$ is **bounded below** provided $(\exists L \in \mathbb{R})\ (\forall n \in \mathbb{N})\ [\ L \leq s_n\ ]$.
⟨such an L is called a lower bound of $\{s_n\}_n$⟩

Def. A sequence $\{s_n\}_n$ is **bounded above** provided $(\exists U \in \mathbb{R})\ (\forall n \in \mathbb{N})\ [\ s_n \leq U\ ]$.
⟨such an U is called a upper bound of $\{s_n\}_n$⟩

Def. A sequence $\{s_n\}_n$ is **bounded** provided $\{s_n\}_n$ is bounded above and bounded below.

Note. A sequence $\{s_n\}_n$ is **bounded** $\Leftrightarrow (\exists M \in \mathbb{R})\ (\forall n \in \mathbb{N})\ [\ |s_n| \leq M\ ] \Leftrightarrow (\exists M \in \mathbb{R}^{>0})\ (\forall n \in \mathbb{N})\ [\ |s_n| \leq M\ ]$.
⟨such an M is called a bound of $\{s_n\}_n$⟩

Thm. Each convergent sequence is bounded.

Common Technique/Useful Lemma

Lemma. Let $a, b \in \mathbb{R}$.

L1. $a = 0 \iff (\forall \varepsilon > 0)\ [\ 0 \leq a < \varepsilon\ ]$. 2.1.9 BS4p28

L2. $a = 0 \iff (\forall \varepsilon > 0)\ [\ 0 \leq a \leq \varepsilon\ ]$. Rmk BS4p28

L3. $a = b \iff (\forall \varepsilon > 0)\ [\ |a - b| < \varepsilon\ ]$.

L4. $a \leq b \iff (\forall \varepsilon > 0)\ [\ a - b < \varepsilon\ ]$.

Algebra of Limits

Thm. Let $c \in \mathbb{R}$ and 3.2.3 BS4p64

$$\lim_{n\to\infty} s_n = S \in \mathbb{R}$$
$$\lim_{n\to\infty} t_n = T \in \mathbb{R}$$

Then the following hold.

A1. $\lim_{n\to\infty} cs_n = c\left(\lim_{n\to\infty} s_n\right)$.

A2. $\lim_{n\to\infty} (s_n + t_n) = \left(\lim_{n\to\infty} s_n\right) + \left(\lim_{n\to\infty} t_n\right)$.

A3. $\lim_{n\to\infty} (s_n - t_n) = \left(\lim_{n\to\infty} s_n\right) - \left(\lim_{n\to\infty} t_n\right)$.

A4. $\lim_{n\to\infty} (s_n \cdot t_n) = \left(\lim_{n\to\infty} s_n\right) \cdot \left(\lim_{n\to\infty} t_n\right)$.

A5. $\lim_{n\to\infty} \frac{s_n}{t_n} = \frac{\lim_{n\to\infty} s_n}{\lim_{n\to\infty} t_n}$, provided $\lim_{n\to\infty} t_n \neq 0$ and $(\forall n)\,[t_n \neq 0]$.

Sequences and Continuous Functions

Def. Let $x_0 \in I \subseteq \mathbb{R}$ with I an opend interval. The function $f \colon I \to \mathbb{R}$ is **continuous at** x_0 provided
$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x \in I)\ [\ |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon\ ]$, or equivalently,
$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x \in I)\ [\ x \in N_\delta(x_0) \implies f(x) \in N_\varepsilon(f(x_0))\ ]$.

Thm. ⟨continuous functions preserve convergent sequences⟩
Let $f \colon I \to \mathbb{R}$ be a continuous fucntion on an open interval I of $\mathbb{R}$.
Let $\lim_{n\to\infty} s_n = L \in I$ and $\{s_n \colon n \in \mathbb{N}\} \subset \mathbb{R}$.
Then $\lim_{n\to\infty} f(s_n) = (L)$.

Order Properties of Limits

Thm. Let $N_0 \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$ and $\{x_n\}_n$ be a sequence from $\mathbb{R}$. Let

$$\lim_{n\to\infty} s_n = S \in \mathbb{R}$$
$$\lim_{n\to\infty} t_n = T \in \mathbb{R}$$

Then the following hold.

O1. If $s_n \le t_n$ for each $n \ge N_0$, then $\lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n$. 3.2.5 BS4p65

O2. If $\alpha \le s_n \le \beta$ for each $n \ge N_0$, then $\alpha \le \lim_{n\to\infty} s_n \le \beta$. 3.2.6 BS4p66

O3. Squeeze Thm. 3.2.7 BS4p66

If $s_n \le x_n \le t_n$ for each $n \ge N_0$ and $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n$,

then $\{x_n\}_n$ also converges and $\lim_{n\to\infty} s_n = \lim_{n\to\infty} x_n = \lim_{n\to\infty} t_n$.

O4. The sequence $\{|s_n|\}_n$ converges and $\lim_{n\to\infty} |s_n| = \left|\left(\lim_{n\to\infty} s_n\right)\right|$. 3.2.9 BS4p68

O5. The sequence $\{\max\{s_n, t_n\}\}_n$ converges and $\lim_{n\to\infty} (\max\{s_n, t_n\}) = \max\left\{\lim_{n\to\infty} s_n, \lim_{n\to\infty} t_n\right\}$.

The sequence $\{\min\{s_n, t_n\}\}_n$ converges and $\lim_{n\to\infty} (\min\{s_n, t_n\}) = \min\left\{\lim_{n\to\infty} s_n, \lim_{n\to\infty} t_n\right\}$.

Helpful : $\max\{s_n, t_n\} = \frac{s_n + t_n}{2} + \frac{|s_n - t_n|}{2}$ and $\min\{s_n, t_n\} = \frac{s_n + t_n}{2} - \frac{|s_n - t_n|}{2}$

I.e., the $\max\{a, b\}$ is (the midpt btw $a\&b$) + (half the distance btw $a\&b$).

while the $\min\{a, b\}$ is (the midpt btw $a\&b$) - (half the distance btw $a\&b$).

Ratio Test for Sequences

Thm. Let $\{x_n\}\, n \in \mathbb{N}$ be a sequence of strictly positive real numbers such that 3.2.11 BS4p69

$$L := \lim_{n\to\infty} \frac{x_{n+1}}{x_n} \qquad \text{(Ratio)}$$

exists and $L < 1$. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges and $\lim_{n\to\infty} x_n = 0$.

Pf. Pf.'s Idea. Find r s.t. $L < r < 1$. Find $N \in \mathbb{N}$ s.t. if $n \ge N$ then $\left|\frac{x_{n+1}}{x_n} - L\right| < (r - L)$.

So if $n \ge N$ then

$$\frac{x_{n+1}}{x_n} < r \qquad \text{and so} \qquad 0 < x_{n+1} < r x_n$$

and so inductively ⟨use math induction to prove⟩ we get

$$0 < x_{n+1} < r^1 x_n < r^1 \left(r^1 x_{n-1}\right) \overset{\text{i.e.}}{=} r^2 x_{n-1} < r^3 x_{n-2} < \cdots < r^{(n-N+1)} x_N \overset{\text{i.e.}}{=} r^n \left[r^{1-N} x_N\right] := C r^n$$

for the constant $C := \left[r^{1-N} x_N\right]$. Note $\lim_{n\to\infty} r^n = 0$ since $0 < r < 1$. Now apply Squeeze Theorem.