

A Rigorous Axiomatic Approach to the Real Numbers

To do analysis, we need a firm grasp on what the real numbers precisely are.

Def. The set of real numbers, denoted \mathbb{R} , is the unique complete ordered field $(F, +, \cdot, <)$.

So in this handout, the set F , with the needed added structure, will end up being \mathbb{R} .

First we define a field as a set which satisfies certain properties called axioms (recall an axiom is a mathematical statement that is accepted without proof). Then we axiomatically (i.e., give the needed properties) define the adjectives ordered and complete. Finally we deal with uniqueness.

1st. Start off with an arbitrary set F . First we put some algebraic structure on F to make it a field.

Field Axioms

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$(F, +, \cdot)$ is a field provided F is a set and the operations (i.e., functions)

$$+ : F \times F \rightarrow F \quad \langle \text{called the addition operator with } +((a, b)) \text{ denoted by } a + b \rangle$$

$$\cdot : F \times F \rightarrow F \quad \langle \text{called the multiplication operator with } \cdot((a, b)) \text{ denoted by } a \cdot b \text{ or } ab \rangle$$

satisfy the following 9 field axioms.

- A1.** For any $a, b \in F$, we have $a + b = b + a$. \langle addition is commutative \rangle
 - A2.** For any $a, b, c \in F$ we have $(a + b) + c = a + (b + c)$. \langle addition is associative \rangle
 - A3.** \exists a unique element $0 \in F$ so that $a + 0 = 0 + a = a$ for all $a \in F$. \langle 0 is called the additive identity. \rangle
 - A4.** $\forall a \in F$ there is an element in F , denoted $-a$, s.t. $a + (-a) = 0$. \langle $-a$ is called the additive inverse of a \rangle
 - M1.** For any $a, b \in F$, we have $ab = ba$. \langle multiplication is commutative \rangle
 - M2.** For any $a, b, c \in F$ we have $(ab)c = a(bc)$. \langle multiplication is associative \rangle
 - M3.** \exists a unique element $1 \in F \setminus \{0\}$ so that $a1 = 1a = a$ for all $a \in F$. \langle 1 is called the multiplicative identity \rangle
 - M4.** $\forall a \in F \setminus \{0\}$ there is an elt. in F , denoted a^{-1} , s.t. $aa^{-1} = 1$. \langle a^{-1} is called the multiplicative inverse of a \rangle
 - AM.** For any $a, b, c \in F$ we have $(a + b)c = ac + bc$. \langle distributive rule, which connects addition and multiplication \rangle
- o. From these nine field axioms, we can show the following.

- (1) The additive inverse of $a \in F$ (see A4) and additive identity 0 (see A3) are unique.
- (2) The multiplicative inverse of $a \in F \setminus \{0\}$ (see M4) and multipl. identity 1 (see M3) are unique.
- (3) $(\forall a \in F) [a \cdot 0 = 0]$.
- (4) If $ab = 0$ then $a = 0$ or $b = 0$.

Thms
2.1.2&
2.1.3

2nd. Now we put an order relation $<$ on the field $(F, +, \cdot)$; thus, creating an ordered field. Recall $<$ is relation on a set F means that $\forall a, b \in F$ the statement $a < b$ is either true or false (but not both).

Order Axioms

$(F, +, \cdot, <)$ is an ordered field provided $(F, +, \cdot)$ is a field and the below 4 axioms hold.

- O1.** For any $a, b \in F$ exactly one of the statements $a = b$, $a < b$, or $b < a$ is true. \langle order trichotomy \rangle
- O2.** For any $a, b, c \in F$ if $a < b$ is true and $b < c$ is true, then $a < c$ is true. \langle transitive/transitivity \rangle
- O3.** For any $a, b, c \in F$ if $a < b$ is true, then $a + c < b + c$ is true.
- O4.** For any $a, b, c \in F$ if $a < b$ is true and $c > 0$ (0 is the additive identity) then $a \cdot c < b \cdot c$ is true.

3rd. Working up to the notion of completeness, let's talk about bounds.

Bounds/Max/Min

Defs. Let $(F, +, \cdot, <)$ be an ordered field and $S \subset F$.

- 1.1. $M \in F$ is an upper bound for S provided if $x \in S$ then $x \leq M$. (Note M need not be in S .)
- 1.2. $M \in F$ is the maximum of S , denoted $M = \max S$, provided M is an upper bound for S and $M \in S$.
- 1.3. S is bounded above provided S has an upper bound in F .
- 2.1. $m \in F$ is a lower bound for S provided if $x \in S$ then $m \leq x$. (Note M need not be in S .)
- 2.2. $m \in F$ is the minimum of S , denoted $m = \min S$, provided m is an lower bound for S and $m \in S$.
- 2.3. S is bounded below provided S has an lower bound in F .
 - o. S is bounded provided S bounded above and bounded below.
 - ?. Here's a question. Does the empty set \emptyset have an upper bound in F ?

4rd. supremum (denoted sup) also called least upper bound (denoted lub) Please see next page.

Completeness Axiom

An ordered field $(F, +, \cdot, <)$ is called complete provided

each nonempty subset S of F that is bounded above has a least upper bound in F

(i.e., if $\emptyset \neq S \subset F$ and S is bounded above, then $\sup S$ exists and $\sup S \in F$).

Fact 1. Existence. (there exists a complete ordered field)

There exists a complete ordered field. The first published rigorous construction of a complete ordered field was done by Dedekind in 1872. Such a construction would be covered in a logic class.

Fact 2. Uniqueness (any two complete ordered fields *mathematically look alike*)

If $(F_1, +, \cdot, <)$ and $(F_2, \oplus, \odot, \otimes)$ are two complete ordered fields, then there exists a bijection (i.e., 1-to-1 and onto function) $f: F_1 \rightarrow F_2$ s.t. for all $x, y \in F_1$

(a) $f(x + y) = f(x) \oplus f(y)$

(b) $f(x \cdot y) = f(x) \odot f(y)$

(c) $x < y$ if and only if $f(x) \otimes f(y)$.

This mapping f is an ordered field isomorphism since f is bijection btw. ordered fields that preserves the field operations (addition&mutiplication) and the order. (Note f tells us that F_1 and F_2 are essentially the same ordered field).

Notation. The unique complete ordered field is called the (set of) real numbers and is denoted \mathbb{R} .

Supremum and Infimum of a Set

Set up

- Let $(F, +, \cdot, <)$ be an ordered field with additive identity 0. (For intuition, think of F being \mathbb{R} .)
- $\widehat{F} := F \cup \{\infty\} \cup \{-\infty\}$ (The symbol $:=$ means define to be equal to. $\widehat{\mathbb{R}}$ is the set of extended real numbers.)
- $S \subset F$

Supremum of S $\stackrel{\text{denoted}}{=} \text{by}$ $\sup S$ is also called Least Upper Bound of S $\stackrel{\text{denoted}}{=} \text{by}$ $\text{lub } S$

Def.'s.

- $\sup S := \infty$ for a nonempty set S that is not bounded above
- $\sup \emptyset := -\infty$. (afterall, the set of upper bounds of \emptyset is all of F)
- Let S be a nonempty set that is bounded above. Then $\beta \in F$ is a sup S provided (sup=l.u.b.)
 - (1) β is an upper bound of S (i.e., $(\forall x \in S) [x \leq \beta]$)
 - (2) β is the least of all the upper bounds of S (i.e., if b is an upper bound of S , then $\beta \leq b$).

Prop. Let S be a nonempty set that is bounded above. Then (2) is equivalent to each of the below.

- (2') if $b < \beta$, then b is not an upper bound of S .
- (2'') if $b < \beta$, then $\exists x_b \in S$ such that $b < x_b$
- (2''') if $\varepsilon > 0$, then $\exists x_\varepsilon \in S$ such that $\beta - \varepsilon < x_\varepsilon$ (here $\varepsilon \in F$)

Thm. Let F be a complete ordered field. Let S be a nonempty set that is bounded above.

Then the $\sup S$ is the unique $\beta \in F$ such that

- (1) β is an upper bound of S (i.e., $\forall x \in S, x \leq \beta$) (existence of an upper bound in F)
- (2) if b is an upper bound of S , then $\beta \leq b$ (uniqueness of the least of the upper bounds)

Summary In a complete ordered field F , the $\sup S \in \widehat{F}$ and

- (1) $\sup S \in F$ if and only if S is nonempty and bounded above
- (2) $\sup S = \infty$ if and only if S is nonempty and not bounded above
- (3) $\sup S = -\infty$ if and only if $S = \emptyset$.

Infimum of S $\stackrel{\text{denoted}}{=} \text{by}$ $\inf S$ also called Greatest Lower Bound of S $\stackrel{\text{denoted}}{=} \text{by}$ $\text{glb } S$

Def.'s.

- $\inf S := -\infty$ for a nonempty set S that is not bounded below.
- $\inf \emptyset := \infty$. (afterall, the set of lower bounds of \emptyset is all of F)
- Let S be a nonempty set that is bounded below. Then $\alpha \in F$ is a inf S provided (inf=g.l.b.)
 - (1) α is a lower bound of S (i.e., $(\forall x \in S) [\alpha \leq x]$)
 - (2) α is the greatest of all the lower bounds of S (if a is a lower bound of S , then $a \leq \alpha$).

Prop. Let S be a nonempty set that is bounded below. Then (2) is equivalent to each of the below.

- (2') if $\alpha < a$, then a is not a lower bound of S .
- (2'') if $\alpha < a$, then $\exists x_a \in S$ such that $x_a < a$
- (2''') if $\varepsilon > 0$, then $\exists x_\varepsilon \in S$ such that $x_\varepsilon < \alpha + \varepsilon$ (here $\varepsilon \in F$)

Thm. Let F be a complete ordered field. Let S be a nonempty set that is bounded below.

Then the $\inf S$ is the unique $\alpha \in F$ such that

- (1) α is a lower bound of S (i.e., $\forall x \in S, \alpha \leq x$) (existence of an lower bound in F)
- (2) if a is a lower bound of S , then $a \leq \alpha$ (uniqueness of the greatest of the lower bounds)

Summary In a complete ordered field F , the $\inf S \in \widehat{F}$ and

- (1) $\inf S \in F$ if and only if S is nonempty and bounded below
- (2) $\inf S = -\infty$ if and only if S is nonempty and not bounded below
- (3) $\inf S = \infty$ if and only if $S = \emptyset$.

Ex 1. Which of the following sets S are fields (resp. ordered fields, complete ordered fields) when endowed with the usual addition operation, multiplication operation, and order?

Fill in the boxes with YES or NO. No proof needed, just use intuition. If answer NO, give a reason.

S ↓	English for S	description of S	field	ordered field	complete ordered field
\mathbb{N}	natural numbers	$\{1, 2, 3, 4, \dots\}$			
\mathbb{Z}	integers	$\{\dots, -2, -1, 0, 1, 2, \dots\}$			
\mathbb{Q}	rational numbers	$\{\frac{a}{b} \in \mathbb{R} : a \in \mathbb{Z}, b \in \mathbb{N}\}$ hint: consider $[0, \sqrt{2}) \cap \mathbb{Q}$			
\mathbb{R}	real numbers	the unique complete ordered field	Yes. by definition	Yes. by definition	Yes. by definition

Ex 2. Consider the below subsets S of \mathbb{R} . Find the following, when they exist. Just use your intuition. No proofs needed. Use \nexists for does not exists. Columns B and E answers may vary. For intuition, think of:

- $\text{lub } S = \text{least upper bound of } S = \text{supremum of } S = \text{sup } S$
- $\text{glb } S = \text{greatest lower bound of } S = \text{infimum of } S = \text{inf } S$.

	order \Rightarrow	D	E	F	H	A	B	C	G
	S ↓	Is S bounded below?	some lower bounds of S	$\min S$	$\text{glb } S$ = $\text{inf } S$	Is S bounded above?	some upper bounds of S	$\max S$	$\text{lub } S$ = $\text{sup } S$
	recall $S \subset \mathbb{R}$	yes/no	must be in \mathbb{R}	must be in S	in $\widehat{\mathbb{R}} :=$ $\mathbb{R} \cup \{\pm\infty\}$	yes/no	must be in \mathbb{R}	must be in S	in $\widehat{\mathbb{R}} :=$ $\mathbb{R} \cup \{\pm\infty\}$
2.1	$\{-3, 2, 5, 7\}$								
2.2	$[0, \sqrt{2}]$								
2.3	$[0, 17)$								
2.4	\mathbb{R}								
2.5	\emptyset								
2.6	$\{x : x^3 < 8\}$								
2.7	$\{\frac{1}{n} : n \in \mathbb{N}\}$								
2.8	$\{\frac{1}{x} : x \in \mathbb{R}^{>0}\}$								
2.9	$\{\cos n\pi : n \in \mathbb{N}\}$								
2.10	$\{n \cos n\pi : n \in \mathbb{N}\}$								
2.11	\mathbb{N}								
2.12	\mathbb{Z}								
2.13	\mathbb{Q}								