A Rigorous Axiomatic Approach to the Real Numbers

To do analysis, we need a firm grasp on what the real numbers precisely are.

Def. The set of real number numbers, denoted $\mathbb{R}$, is the unique complete ordered field. TBB p4

First we will define a field as a set which satisfies certain properties (called axioms ⟨recall an axiom is a mathematical statement that is accepted without proof⟩). Then we will axiomically (i.e., give the needed properties) define the adjectives ordered and complete. Finally we deal with uniquness.

1^{st}. Start off with an arbitrary set F. First we put some structure on F to make it a field.

Field Axioms $(F, +, \cdot)$ is a field provided F is a set and the operations (i.e., functions) TBB §1.3 Math546 Math547

$$+ \colon F \times F \to F \quad \langle \text{called the addition operator with } + ((a,b)) \text{ denoted by } a+b \rangle$$

$$\cdot \colon F \times F \to F \quad \langle \text{called the multiplication operator with } \cdot\ ((a,b)) \text{ denoted by } a \cdot b \text{ or } ab \rangle$$

satisfy the following 9 field axioms.

A1. For any $a, b \in F$, we have $a + b = b + a$. ⟨addition is commutative⟩

A2. For any $a, b, c \in F$ we have $(a + b) + c = a + (b + c)$. ⟨addition is associative⟩

A3. $\exists$ a unique element $0 \in F$ so that $a + 0 = 0 + a = a$ for all $a \in F$. ⟨0 is called the additive identity.⟩

A4. $\forall a \in F$ there is an element in F, denoted $-a$, s.t. $a + (-a) = 0$. ⟨$-a$ is called the additive inverse of a⟩

M1. For any $a, b \in F$, we have $ab = ba$. ⟨multiplication is commutative⟩

M2. For any $a, b, c \in F$ we have $(ab)c = a(bc)$. ⟨multiplication is associative⟩

M3. $\exists$ a unique element $1 \in F \setminus \{0\}$ so that $a1 = 1a = a$ for all $a \in F$. ⟨1 is called the multiplicative identity⟩

M4. $\forall a \in F \setminus \{0\}$ there is an elt. in F, denoted a^{-1}, s.t. $aa^{-1} = 1$. ⟨a^{-1} is called the multiplicative inverse of a⟩

AM1. For any $a, b, c \in F$ we have $(a + b)\,c = ac + bc$. ⟨distributive rule, which connects addition and multiplication⟩

2^{nd}. Now we put an order relation $<$ on the field $(F, +, \cdot)$; thus, creating an ordered field. Recall $<$ is relation on a set F means that $\forall a, b \in F$ the statement $a < b$ is either true or false (but not both). TBB Appendix A4

Order Axioms $(F, +, \cdot, <)$ is an ordered field provided $(F, +, \cdot)$ is a field and the below 4 axioms hold. TBB §1.4

O1. For any $a, b \in F$ exactly one of the statements $a = b$, $a < b$, or $b < a$ is true. ⟨order trichotomy⟩

O2. For any $a, b, c \in F$ if $a < b$ is true and $b < c$ is true, then $a < c$ is true. ⟨transitive/transitivity⟩

O3. For any $a, b, c \in F$ if $a < b$ is true, then $a + c < b + c$ is true.

O4. For any $a, b, c \in F$ if $a < b$ is true and $c > 0$ ⟨0 is the additive identity⟩ then $a \cdot c < b \cdot c$ is true.

3^{rd}. Bounds/Max/Min Working up to the notion of completeness, let's talk about bounds. TBB §1.5 TBB §1.6

Defs. Let $(F, +, \cdot, <)$ be an ordered field and $S \subset F$.

1.1. $M \in F$ is an upper bound for S provided if $x \in S$ then $x \leq M$. ⟨Note M need not be in S.⟩

1.2. $M \in F$ is the maximum of S, denoted $M = \max S$, provided M is an upper bound for S and $M \in S$.

1.3. S is bounded above provided S has an upper bound in F.

2.1. $m \in F$ is a lower bound for S provided if $x \in S$ then $m \leq x$. ⟨Note M need not be in S.⟩

2.2. $m \in F$ is the minimum of S, denoted $m = \min S$, provided m is an lower bound for S and $m \in S$

2.3. S is bounded below provided S has an lower bound in F.

◦. S is bounded provided S bounded above and bounded below.

?. Here's a question. Does the empty set $\emptyset$ have an upper bound in F?

4^{rd}. supremum ⟨denoted sup⟩ also called least upper bound ⟨denoted lub⟩ Please see next page. TBB §1.6

Completeness Axiom An ordered field $(F, +, \cdot, <)$ is called complete provided each nonempty subset S of F that is bounded above has a least upper bound in F (i.e., if $\emptyset \neq S \subset F$ and S is bounded above, then $\sup S$ exists and $\sup S \in F$).

Fact 1. There exists a complete ordered field. The first published rigorous construction of a complete ordered field was done by Dedekind in 1872. Such a construction would be covered in a logic class.

Fact 2. If $(F_1, +, \cdot, <)$ and $(F_2, \bigoplus, \bigodot, \bigotimes)$ are two complete ordered fields, then there exists a bijection (i.e., 1-to-1 and onto function) $f \colon F_1 \to F_2$ s.t. for all $x, y \in F_1$ TBB Exercise 1.11.3

(a) $f(x + y) = f(x) \bigoplus f(y)$

(b) $f(x \cdot y) = f(x) \bigodot f(y)$

(c) $x < y$ if an only if $f(x) \otimes f(y)$.

This mapping f is an ordered field isomorphism since f is bijection btw. ordered fields that preserves the field operations (addition&mutiplication) and the order. ⟨Note f tells us that F_1 and F_2 are essentially the same ordered field⟩.

Supremum and Infimum of a Set

Set up

- Let $(F, +, \cdot, <)$ be an ordered field with additive identity 0. ⟨For intuition, think of F being $\mathbb{R}$.⟩
- $\widehat{F} := F \cup \{\infty\} \cup \{-\infty\}$ ⟨The symbol := means define to be equal to. $\widehat{\mathbb{R}}$ is the set of extended real numbers.⟩
- $S \subset F$

Supremum of S $\overset{\text{denoted}}{\underset{\text{by}}{=}}$ $\sup S$ is also called Least Upper Bound of S $\overset{\text{denoted}}{\underset{\text{by}}{=}}$ lub S

Def.'s.

- $\sup S := \infty$ for a nonempty set S that is not bounded above
- $\sup \emptyset := -\infty$. ⟨afterall, the set of upper bounds of $\emptyset$ is all of F⟩
- Let S be a nonempty set that is bounded above. Then $\beta \in F$ is a $\sup S$ provided
  (1) β is an upper bound of S (i.e., $\forall x \in S$, $x \le \beta$)
  (2) if b is an upper bound of S, then $\beta \le b$ ⟨i.e., β is the least of all the upper bounds of S⟩.

Prop. Let S be a nonempty set that is bounded above. Then (2) is equivalent to each of the below.

($2'$) if $b < \beta$, then b is not an upper bound of S .
($2''$) if $b < \beta$, then $\exists\, x_b \in S$ such that $b < x_b$
($2'''$) if $\varepsilon \in F$ and $\varepsilon > 0$, then $\exists\, x_\varepsilon \in S$ such that $\beta - \varepsilon < x_\varepsilon$

Thm. Let F be a complete ordered field. Let S be a nonempty set that is bounded above. Then the $\sup S$ is the unique $\beta \in F$ such that

(1) β is an upper bound of S (i.e., $\forall x \in S$, $x \le \beta$) ⟨existence of an upper bound in F⟩
(2) if b is an upper bound of S, then $\beta \le b$ ⟨uniqueness of the least of the upper bounds⟩

Summary In a complete ordered field F, the $\sup S \in \widehat{F}$ and

(1) $\sup S \in F$ if and only if S is nonempty and bounded above
(2) $\sup S = \infty$ if and only if S is nonempty and not bounded above
(3) $\sup S = -\infty$ if and only if $S = \emptyset$.

Infimum of S $\overset{\text{denoted}}{\underset{\text{by}}{=}}$ $\inf S$ also called Greatest Lower Bound of S $\overset{\text{denoted}}{\underset{\text{by}}{=}}$ glb S

Def.'s.

- $\inf S := -\infty$ for a nonempty set S that is not bounded below.
- $\inf \emptyset := \infty$. ⟨afterall, the set of lower bounds of $\emptyset$ is all of F⟩
- Let S be a nonempty set that is bounded below. Then $\alpha \in F$ is an $\inf S$ provided
  (1) α is a lower bound of S (i.e., $\forall x \in S$, $\alpha \le x$)
  (2) if a is a lower bound of S, then $a \le \alpha$ ⟨i.e., α is the greatest of all the lower bounds of S⟩.

Prop. Let S be a nonempty set that is bounded below. Then (2) is equivalent to each of the below.

($2'$) if $\alpha < a$, then a is not a lower bound of S.
($2''$) if $\alpha < a$, then $\exists\, x_a \in S$ such that $x_a < a$
($2'''$) if $\varepsilon \in F$ and $\varepsilon > 0$, then $\exists\, x_\varepsilon \in S$ such that $x_\varepsilon < \alpha + \varepsilon$

Thm. Let F be a complete ordered field. Let S be a nonempty set that is bounded below. Then the $\inf S$ is the unique $\alpha \in F$ such that

(1) α is a lower bound of S (i.e., $\forall x \in S$, $\alpha \le x$) ⟨existence of an lower bound in F⟩
(2) if a is a lower bound of S, then $a \le \alpha$ ⟨uniqueness of the greatest of the lower bounds⟩

Summary In a complete ordered field F, the $\inf S \in \widehat{F}$ and

(1) $\inf S \in F$ if and only if S is nonempty and bounded below
(2) $\inf S = -\infty$ if and only if S is nonempty and not bounded below
(3) $\inf S = \infty$ if and only if $S = \emptyset$.

Ex 1. Which of the following sets S are fields (resp. ordered fields, complete ordered fields) when endowed with the usual addition operation, muliplication operation, and order?

Fill in the boxes with YES or NO. No proof needed, just use intuition. If answer NO, give a reason.

S $\Downarrow$	English for S	description of S	field	ordered field	complete ordered field
$\mathbb{N}$	natural numbers	$\{1, 2, 3, 4, \ldots\}$			
$\mathbb{Z}$	integers	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$			
$\mathbb{Q}$	rational numbers	$\left\{\frac{a}{b} \in \mathbb{R} \colon a \in \mathbb{Z}, b \in \mathbb{N}\right\}$ hint: consider $\left[0, \sqrt{2}\right) \cap \mathbb{Q}$			
$\mathbb{R}$	real numbers	the unique complete ordered field	Yes. by definition	Yes. by definition	Yes. by definition

Ex 2. Consider the below subsets S of $\mathbb{R}$. Find the following, when they exist. Just use your intuition. No proofs needed. Use $\nexists$ for does not exists. Columns B and E answers may vary. For intuition, think of: TBB 1.6.2

- ∘ lub S = least upper bound of S = supremum of S = $\sup S$
- ∘ glb S = greatest lower bound of S = infimum of S = $\inf S$.

	order $\Rightarrow$	D	E	F	H	A	B	C	G
	S $\Downarrow$	Is S bounded below?	some lower bounds of S	$\min S$	glb S = $\inf S$	Is S bounded above?	some upper bounds of S	$\max S$	lub S = $\sup S$
	recall $S \subset \mathbb{R}$	yes/no	must be in $\mathbb{R}$	must be in S	in $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$	yes/no	must be in $\mathbb{R}$	must be in S	in $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$
2.1	$\{-3, 2, 5, 7\}$								
2.2	$[\,0, \sqrt{2}\,]$								
2.3	$[\,0, 17\,)$								
2.4	$\mathbb{R}$								
2.5	$\emptyset$								
2.6	$\{x \colon x^3 < 8\}$								
2.7	$\left\{\frac{1}{n} \colon n \in \mathbb{N}\right\}$								
2.8	$\left\{\frac{1}{x} \colon x \in \mathbb{R}^{>0}\right\}$								
2.9	$\{\cos n\pi \colon n \in \mathbb{N}\}$								
2.10	$\{n \cos n\pi \colon n \in \mathbb{N}\}$								
2.11	$\mathbb{N}$								
2.12	$\mathbb{Z}$								
2.13	$\mathbb{Q}$								