Summary $\overline{\lim}$ and $\underline{\lim}$.

Have bnd seq. $\{x_n\}_n$.

Def $\overline{\lim}\, x_n$. $s_n \stackrel{\text{def}}{:=} \sup\{x_k : k \geq n\}$. $s_n \searrow \left(\overline{\lim\limits_{n\to\infty}}\, x_n\right)$.

Def $\underline{\lim}\, x_n$. $t_n \stackrel{\text{def}}{:=} \inf\{x_k : k \geq n\}$. $t_n \nearrow \left(\underline{\lim\limits_{n\to\infty}}\, x_n\right)$.

Thm 3.3.A Let $\{x_n\}$ be a bounded sequence st. $\overline{\lim\limits_{n\to\infty}}\, x_n = \underline{\lim\limits_{n\to\infty}}\, x_n$.

Then $\{x_n\}$ converges and $\lim\limits_{n\to\infty} x_n = \overline{\lim\limits_{n\to\infty}}\, x_n = \underline{\lim\limits_{n\to\infty}}\, x_n$.

Thm 3.4.2 If $\lim\limits_{n\to\infty} x_n = x$ and $\{x_{n_k}\}_k$ is a subseq of $\{x_n\}_n$,

then $\{x_{n_k}\}_k$ conv. to x., i.e. $\lim\limits_{k\to\infty} x_{n_k} = x$.

Thm 3.4.51 Let $\{x_n\}$ be a bounded sequence,

1. $\exists$ subseq. $\{x_{n_k}\}_{k=1}^{\infty}$ st. $\lim\limits_{k\to\infty} x_{n_k} = \overline{\lim\limits_{n\to\infty}}\, x_n$,
2. $\exists$ another subseq. $\{x_{n_k}\}_{k=1}^{\infty}$ st. $\lim\limits_{k\to\infty} x_k = \underline{\lim\limits_{n\to\infty}}\, x_n$.

Thm 3.4.N Let $\{x_n\}$ be a bounded sequence. Then

$$\overline{\lim_{n\to\infty}}\,(-x_n) = -\underline{\lim_{n\to\infty}}\, x_n.$$

Thm 3.4.52 Let $\{x_n\}$ be a bnd seq. Then

$$\left[\begin{array}{c}\{x_n\} \text{ converges} \\ \text{and} \\ \lim\limits_{n\to\infty} x_n = L\end{array}\right] \iff \left[\begin{array}{c}\overline{\lim\limits_{n\to\infty}}\, x_n = \underline{\lim\limits_{n\to\infty}}\, x_n \\ \text{and} \\ L = \overline{\lim\limits_{n\to\infty}}\, x_n\end{array}\right]$$

Thm 3.4.8 Bolzano-Weierstrass Thm

A bounded sequence has a convergent subsequence.