Defs. (M,d) is a metric space provided M is a nonempty set and d is a metric on M. TBB §13.2
Note, d is a metric on M provided $d\colon M \times M \to \mathbb{R}$ is a function that satisfies, for each $x,y,z \in M$,

(M1) $d(x,y) \geq 0$ (M3) $d(x,y) = d(y,x)$ (symmetric)
(M2) $d(x,y) = 0$ if and only if $x = y$ (M4) $d(x,y) \leq d(x,z) + d(z,y)$. (triangle inequality)

If d is understood, often we refer to (M,d) by just M. HMWK: read §13.1–13.3.

►. Throughout this handout, (M,d) is a metric space (e.g., $M = \mathbb{R}$ with $d(x,y) := |x-y|$) and

$$S, G, F, K \subset M \qquad \text{and} \qquad x, x_0, y \in M \qquad \text{and} \qquad \varepsilon > 0$$

and def stands for definition while NTN stands for notation.

Neighborhood (NBHD) §4.2.1

$N_\varepsilon(x_0) \overset{\text{NTN}}{=}$ ε-NBHD centered at x_0 $\overset{\text{def}}{=} \{y \in M\colon d(y,x_0) < \varepsilon\}$

$N'_\varepsilon(x_0) \overset{\text{NTN}}{=}$ deleted ε-NBHD centered at x_0 $\overset{\text{def}}{=} \{y \in M\colon 0 < d(y,x_0) < \varepsilon\} = N_\varepsilon(x_0) \setminus \{x_0\}$

In the TB[2] book, if $x \in N_\varepsilon(x_0)$ then $N_\varepsilon(x_0)$ is called a NBHD of x. We will avoid this terminology. In reality, a NBHD of x is any open set containing x.

Neighborhood (NBHD) for $M = \mathbb{R}$ with $d(x,y) := |x-y|$

$$N_\varepsilon(x_0) = \{y \in \mathbb{R}\colon |x_0 - y| < \varepsilon\} = (x_0 - \varepsilon, x_0 + \varepsilon)$$
$$N'_\varepsilon(x_0) = \{y \in \mathbb{R}\colon 0 < |x_0 - y| < \varepsilon\} = (x_0 - \varepsilon, x_0) \cup (x_0, x_0 + \varepsilon)$$

DEFINITIONS AND NOTATION

x_0 is an interior point of S $\overset{\text{NTN}}{\Longleftrightarrow}$ $x_0 \in S^o$ $\overset{\text{def}}{\Longleftrightarrow}$ $(\exists \varepsilon > 0)\ [N_\varepsilon(x_0) \subset S]$ $\underset{\text{book}}{\overset{\text{NTN}}{\Longleftrightarrow}}$ $x_0 \in \text{int}\,(S)$ §4.2.1

x_0 is a limit point[1] of S $\overset{\text{NTN}}{\Longleftrightarrow}$ $x_0 \in S'$ $\overset{\text{def}}{\Longleftrightarrow}$ $(\forall \varepsilon > 0)\ [N'_\varepsilon(x_0) \cap S \neq \emptyset]$ §4.2.3
$\overset{\text{i.e.}}{\Longleftrightarrow}$ $(\forall \varepsilon > 0)\ (\exists y \in S)\ [y \in N'_\varepsilon(x_0)]$

x_0 is an isolated point of S $\overset{\text{NTN}}{\Longleftrightarrow}$ none $\overset{\text{def}}{\Longleftrightarrow}$ $[x_0 \in S]$ and $[(\exists \varepsilon > 0)\ [N'_\varepsilon(x_0) \cap S = \emptyset]]$ §4.2.2
$\Longleftrightarrow$ $(\exists \varepsilon > 0)\ [N_\varepsilon(x_0) \cap S = \{x_0\}]$

x_0 is a boundary point of S $\overset{\text{NTN}}{\Longleftrightarrow}$ $x_0 \in \partial S$ $\overset{\text{def}}{\Longleftrightarrow}$ $(\forall \varepsilon > 0)\ \left[N_\varepsilon(x_0) \cap S \neq \emptyset \text{ and } N_\varepsilon(x_0) \cap S^C \neq \emptyset\right]$ §4.2.4

the interior of S $\overset{\text{NTN}}{=}$ S^o $\overset{\text{def}}{=}$ the set of all interior points of S §4.3

the closure of S $\overset{\text{NTN}}{=}$ $\overline{S}$ $\overset{\text{def}}{=}$ $S \cup S'$ §4.3

the boundary of S $\overset{\text{NTN}}{=}$ ∂S $\overset{\text{def}}{=}$ the set of all boundary points of S

SEQUENTIAL CHARACTERIZATIONS

A sequence $\{x_n\}_{n=1}^\infty$ from M converges to x provided $(\forall \varepsilon > 0)\ (\exists N \in \mathbb{N})\ (\forall n \in \mathbb{N})\ [n \geq N \Rightarrow x_n \in N_\varepsilon(x_0)]$.
$x_0 \in S'$ if and only if there is a sequence $\{s_n\}_{n=1}^\infty$ from S such that $\lim_{n\to\infty} s_n = x_0$ and, $\forall n \in \mathbb{N}$, $s_n \neq x_0$.
$x_0 \in \overline{S}$ if and only if there is a sequence $\{s_n\}_{n=1}^\infty$ from S such that $\lim_{n\to\infty} s_n = x_0$.

DEFINITION OF OPEN AND CLOSED SET

G is open $\overset{\text{def}}{\Longleftrightarrow}$ each point in G is an interior point of G $\overset{\text{i.e.}}{\Longleftrightarrow}$ $(\forall x \in G)\ (\exists \varepsilon > 0)\ [N_\varepsilon(x) \subset G]$ §4.3.2

F is closed[2] $\overset{\text{def}}{\Longleftrightarrow}$ $F^C \overset{\text{def}}{=} M \setminus F$ is an open set

PROPOSITIONS (follow directly from defs.)

- S is closed $\overset{\text{thm}}{\Longleftrightarrow}$ S contains all its limit points $\overset{\text{i.e.}}{\Longleftrightarrow}$ $S' \subset S$.
- (isolated point of S) $\subset S \subset$ (isolated point of S) $\uplus S'$ where $\uplus$ means disjoint union.
- $\left(\overline{S}\right)' \subset S'$ (and so $\left(\overline{S}\right)' \subset S' \subset S \cup S' = \overline{S}$ thus $\overline{S}$ is closed)

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[1] Another word for limit point is accumulation point.

[2] We will use this def. of closed and not the book's def.!!! The two definitions are equivalent but ours is more widely used. See book's Thm. 4.16 (S is closed $\Leftrightarrow$ S^C is open), Def. 4.9 (S is closed $\Leftrightarrow$ $S' \subset S$), and next fact.

UNIONS AND INTERSECTION OF OPEN/CLOSED SETS

§4.4

Thm. Let Γ be an arbitrary indexing set and $n \in \mathbb{N}$.

- If $\{G_\gamma\}_{\gamma\in\Gamma}$ and $\{G_i\}_{i=1}^n$ are collections of open subsets of a metric space M, then:

$$\bigcup_{\gamma\in\Gamma} G_\gamma \text{ is open} \qquad \text{and} \qquad \bigcap_{i=1}^{n} G_i \text{ is open.}$$

- If $\{F_\gamma\}_{\gamma\in\Gamma}$ and $\{F_i\}_{i=1}^n$ are collections of closed subsets of a metric space M, then:

$$\bigcap_{\gamma\in\Gamma} F_\gamma \text{ is closed} \qquad \text{and} \qquad \bigcup_{i=1}^{n} F_i \text{ is closed.}$$

Recall. $x \in \bigcup_{\gamma\in\Gamma} G_\gamma \overset{\text{def}}{\Longleftrightarrow} (\exists \gamma \in \Gamma)\ [x \in G_\gamma]$ while $x \in \bigcap_{\gamma\in\Gamma} F_\gamma \overset{\text{def}}{\Longleftrightarrow} (\forall \gamma \in \Gamma)\ [x \in F_\gamma]$

One Theorem for when (M, d) is the $\mathbb{R}$ with $d(x,y) := |x - y|$.

Thm. A subset G of $\mathbb{R}$ is open if and only if
the set $G = \bigcup_{n=1}^{\infty} (a_n, b_n)$ for some disjoint open (possibly degenerate/empty) intervals $\{(a_n, b_n)\}_{n=1}^{\infty}$.

INTERIOR OF A SUBSET S OF A METRIC SPACE

Order to show: 1, 5, 2, 3, 6, 4.

(1) interior of $S \overset{\text{NTN}}{=} S^o \overset{\text{def}}{=}$ set of interior points of S

(2) S^o is open

(3) $S^o \subset S$

(4) $S^o = S \ \Leftrightarrow\ S$ is open

(5) $S^o = \bigcup_{G\in\mathcal{G}_S} G$ where $\mathcal{G}_S = \{G \in \mathcal{P}(M) : G \text{ is open and } G \subset S\}$.

(6) S^o is the largest open set contained in S (i.e., S^o is the largest open set *inside of* S) in the sense that S^o is an open set contained in S and ⟨now the largest part⟩ if H is an open set contained in S then $H \subset S^o$.

CLOSURE OF A SUBSET S OF A METRIC SPACE

Order to show: 1, 3, 2, 4, 5, 6.

(1) closure of $S \overset{\text{NTN}}{=} \overline{S} \overset{\text{def}}{=} S \cup \partial S \overset{\text{thm}}{=} S \cup S'$

(2) $\overline{S}$ is closed

(3) $S \subset \overline{S}$

(4) $S = \overline{S} \ \Leftrightarrow\ S$ is closed

(5) $\overline{S} = \bigcap_{F\in\mathcal{F}_S} F$ where $\mathcal{F}_S = \{F \in \mathcal{P}(M) : F \text{ is closed and } S \subset F\}$.

(6) $\overline{S}$ is the smallest closed set that contains S (i.e., $\overline{S}$ is the smallest closed set that *sits on top of* S) i.e., $\overline{S}$ is a closed set that contains S and ⟨now the smallest part⟩ if H is an closed set that contains S then $\overline{S} \subset H$.

COMPACT SUBSETS OF A METRIC SPACE §4.5

Defs. A collection

$$\mathcal{C} \;=\; \{G_\gamma\}_{\gamma\in\Gamma}$$

of subsets of M is an <u>OPEN COVERING</u> of S provided each G_γ is open and the G_γ's *cover* S in the sense that

$$S \subset \bigcup_{\gamma\in\Gamma} G_\gamma \;.$$

We call $\widetilde{\mathcal{C}}$ a <u>FINITE SUBCOVERING</u> of S (of the covering $\mathcal{C}$) provided, for some $n \in \mathbb{N}$,

$$\widetilde{\mathcal{C}} \;=\; \{G_{\gamma_i}\}_{i=1}^{n} \;\subset\; \mathcal{C} \qquad\text{and}\qquad S \subset \bigcup_{i=1}^{n} G_{\gamma_i}.$$

Def. A subset K of M is <u>COMPACT</u> [3] provided each open covering of K has a finite subcovering of K. So:

$$K \text{ is compact} \iff \forall \text{ open covering } \mathcal{C} \text{ of } K \;\exists \text{ finite subcovering } \widetilde{\mathcal{C}} \text{ of } K\;.$$

Lem. Lemmata towards the Heine-Borel Thm.

L1. A compact subset of $\mathbb{R}$ is bounded.

L2. A compact subset of a metric space is closed.

L3. A closed subset of a compact set in a metric space is compact.

L4. A closed and bounded interval of $\mathbb{R}$ is compact.

Rest of Handout (M,d) is the $\mathbb{R}$ with $d(x,y) := |x-y|$.

HEINE-BOREL THEOREM

Thm. Let $S \subset \mathbb{R}$. Each open covering of S has a finite subcovering if and only if S is closed and bounded. I.e.,

$$\text{a subset } S \text{ of } \mathbb{R} \text{ is compact} \quad\Leftrightarrow\quad S \text{ is closed and bounded}\;.$$

BW = BOLZANO-WEIERSTRASS

Thm. Recall the (baby) BW Thm. Each bounded sequence from $\mathbb{R}$ contains a convergent subsequence. Thm2.40

Thm. BW Thm.[4] (sequential form, $\mathrm{BWP}_{\mathrm{seq}}$) Thm4.21

a subset S of $\mathbb{R}$ is compact $\Leftrightarrow$ each sequence from S has a subseq. that converges to a point in S.

Thm. BW Thm. (set form, $\mathrm{BWP}_{\mathrm{set}}$) Cor4.22

a subset S of $\mathbb{R}$ is compact $\Leftrightarrow$ each infinite subset of S has at least one limit point that is in S .

NESTED SETS

Def. The <u>diameter</u> of a subset E of a metric space (M,d) is $\operatorname{diam} E := \sup\{d(x,y) : x,y \in E\}$.

Thm. Recall the Nested Interval Property. If a sequence $\{\,[a_n,b_n]\,\}_{n=1}^{\infty}$ of nonempty closed interval of $\mathbb{R}$ satisfying ER2.9.6

$$[a_n,b_n] \supset [a_{n+1},b_{n+1}] \quad \text{for each } n\in\mathbb{N} \qquad\text{and}\qquad \lim_{n\to\infty} \operatorname{diam}\,[a_{n+1},b_{n+1}] = 0$$

then $\cap_{n=1}^{\infty} I_n$ contains precisely one point.

Thm. If a sequence $\{E_n\}_{n=1}^{\infty}$ of nonempty compact subsets of $\mathbb{R}$ satisfies Thm4.24

$$E_n \supset E_{n+1} \quad \text{for each } n \in \mathbb{N}$$

then $\cap_{n=1}^{\infty} E_n$ is nonempty

Thm. Cantor Intersection Thnm. If a sequence $\{E_n\}_{n=1}^{\infty}$ of compact closed subsets of $\mathbb{R}$ satisfies Thm4.25

$$E_n \supset E_{n+1} \quad \text{for each } n\in\mathbb{N} \qquad\text{and}\qquad \lim_{n\to\infty} \operatorname{diam} E_n = 0$$

then $\cap_{n=1}^{\infty} E_n$ contains precisely one point.

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[3] We will use this def. of compact and not the book's def.!!! Our def. of compact is the correct topological def. and is equivalent to, in the special case that $M = \mathbb{R}$ (with the usual metric), the book's def.'s Def. 4.34.

[4] TBB book calls this theorem the Bolzano-Weierstrass Property (see Thm. 4.21).

For this chart, let the metric space $(M, d) = (\mathbb{R}, d)$ where d is the usual metric on $\mathbb{R}$, $d(x, y) = |x - y|$. Fill in the chart. No explanation required. Here, $S \subset \mathbb{R}$ and $a, b \in \mathbb{R}$ with $a < b$. Recall:

- x_0 is an interior point of S $\overset{\text{def}}{\Longleftrightarrow}$ $(\exists \varepsilon > 0)\ [\, N_\varepsilon(x_0) \subset S\,]$
- x_0 is a limit point of S $\overset{\text{def}}{\Longleftrightarrow}$ $(\forall \varepsilon > 0)\ [\, N'_\varepsilon(x_0) \cap S \neq \emptyset\,] \overset{\text{i.e.}}{\Longleftrightarrow} (\forall \varepsilon > 0)\ (\exists y \in S)\ [y \in N'_\varepsilon(x_0)]$.
- $\overline{S} \overset{\text{def}}{=} S \cup S'$.
- $x_0 \in S' \Leftrightarrow$ there is a sequence $\{s_n\}_{n=1}^{\infty}$ from S such that $\lim\limits_{n\to\infty} s_n = x_0$ and, $\forall n \in \mathbb{N}$, $s_n \neq x_0$.
- $x_0 \in \overline{S} \Leftrightarrow$ there is a sequence $\{s_n\}_{n=1}^{\infty}$ from S such that $\lim\limits_{n\to\infty} s_n = x_0$.

S	interior of S S^o	limit points of S S'	closure of S $\overline{S}$	Is S open? yes/no	Is S closed? yes/no
$(a, b]$	(a, b)	$[a, b]$	$[a, b]$	no	no
(a, b)	(a, b)	$[a, b]$	$[a, b]$	yes	no
$[a, b]$	(a, b)	$[a, b]$	$[a, b]$	no	yes
(a, ∞)	(a, ∞)	$[a, \infty)$	$[a, \infty)$	yes	no
$(0, 1) \cup \{17\}$	$(0, 1)$	$[0, 1]$	$[0, 1] \cup \{17\}$	no	no
$\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$	$\emptyset$	$\{0\}$	$\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$	no	no
$\mathbb{Q}$	$\emptyset$	$\mathbb{R}$	$\mathbb{R}$	no	no
$[0, 1] \cap \mathbb{Q}$	$\emptyset$	$[0, 1]$	$[0, 1]$	no	no
$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	yes	yes
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	yes	yes