►. **Set up for this entire handout**. Let $f\colon X \to Y$ be a function from a set X to a set Y. Let $x \in X$ and $y \in Y$. Let $A \subseteq X$ and $B \subseteq Y$. Pictorially have

$$\begin{array}{cccc} f\colon & X & \to & Y \\ & \cup & & \cup \\ & A & & B\,. \end{array}$$

Defs. The function f is injective (or one-to-one) provided if $x_1, x_2 \in X$ and $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.
⟨So f is 1-to-1 $\Leftrightarrow$ [if $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$, then $x_1 = x_2$.]⟩
The function f is surjective (or onto) provided if $y \in Y$ then there is $x \in X$ such that $f(x) = y$.
The function f is bijective provided f is injective and surjective.
⟨The function $f\colon X \to Y$ has and inverse function $f^{-1}\colon Y \to X$ if and only if f is bijective.⟩
The image (or direct image) of A under f, denoted by $f[A]$, is the set

$$f[A] \overset{\text{def}}{:=} \{f(x) \in Y : x \in A\} \overset{\text{note}}{=} \{y \in Y : f(x) = y \text{ for some } x \in A\} \overset{\text{note}}{\subseteq} Y\,.$$

The inverse image (or preimage) of B under f, denoted by $f^{-1}[B]$, is the set

$$f^{-1}[B] \overset{\text{def}}{:=} \{x \in X : f(x) \in B\} \overset{\text{note}}{=} \{x \in X : f(x) = y \text{ for some } y \in B\} \overset{\text{note}}{\subseteq} X\,.$$

Rmk.

$$y \in f[A] \quad \underset{\text{of image}}{\overset{\text{by def.}}{\Longleftrightarrow}} \quad y = f(a) \text{ for some } a \in A \quad \overset{\text{note}}{\Longleftrightarrow} \quad (\exists a \in A)\ [f(a) = y]$$

$$x \in f^{-1}[B] \quad \underset{\text{of preimage}}{\overset{\text{by def.}}{\Longleftrightarrow}} \quad f(x) \in B\,.$$

Be careful though:

$$x \in A \quad \Rightarrow \quad f(x) \in f(A)$$

$$f(x) \in f(A) \quad \not\Rightarrow \quad x \in A. \qquad \text{(W)}$$

Proposition. Let $A_1, A_2, A_i \subseteq X$ and $B_1, B_2, B_i \subseteq Y$ for each i in an indexing set I. Then

$$x \in \bigcup_{i \in I} A_i \quad \underset{\text{of union}}{\overset{\text{by def}}{\Longleftrightarrow}} \quad x \in A_i \text{ for some } i \in I \quad \overset{\text{note}}{\Longleftrightarrow} \quad (\exists i \in I)\ [x \in A_i]$$

$$x \in \bigcap_{i \in I} A_i \quad \underset{\text{of intersection}}{\overset{\text{by def}}{\Longleftrightarrow}} \quad x \in A_i \text{ for each } i \in I \quad \overset{\text{note}}{\Longleftrightarrow} \quad (\forall i \in I)\ [x \in A_i]$$

and

$$A_1 \subseteq A_2 \quad \Rightarrow \quad f[A_1] \subseteq f[A_2] \qquad (1)$$

$$B_1 \subseteq B_2 \quad \Rightarrow \quad f^{-1}[B_1] \subseteq f^{-1}[B_2] \qquad (2)$$

and

$$A \subseteq f^{-1}[f[A]] \qquad (3)$$

$$f\left[f^{-1}[B]\right] \subseteq B \qquad (4)$$

and

$$f[\cup_{i \in I} A_i] = \cup_{i \in I} f[A_i] \qquad (5)$$

$$f[\cap_{i \in I} A_i] \subseteq \cap_{i \in I} f[A_i] \qquad (6)$$

$$f[X \setminus A] \supseteq f[X] \setminus f[A] \qquad (7)$$

$$f^{-1}[\cup_{i \in I} B_i] = \cup_{i \in I} f^{-1}[B_i] \qquad (8)$$

$$f^{-1}[\cap_{i \in I} B_i] = \cap_{i \in I} f^{-1}[B_i]\,. \qquad (9)$$

$$f^{-1}[Y \setminus B] = X \setminus f^{-1}[B] \qquad \langle \text{i.e. } f^{-1}\left[B^C\right] = \left(f^{-1}[B]^C\right) \rangle\,. \qquad (10)$$

Remarks:

- If f is injective (i.e., 1-to-1), then equality holds in (3) & (7) and the implication in (W) holds.
- If f is surjective (i.e., onto), then set equality holds in (4).
- To help remember (6), think about what can happen when $\cap_{i \in I} A_i = \emptyset$.