§1.3. Part 1. Equivalence Relation

Ex 1 On $\mathbb{R}$, the operation $=$ (i.e. "is equal to") is a *relation* on $\mathbb{R}$.

Def. The operation $\sim$ is a *relation on a set X* provided if $x_1, x_2 \in X$, then

either $x_1 \sim x_2$ (i.e. x_1 tidles x_2)
or $x_1 \nsim x_2$. (i.e. x_1 does not tidle x_2)

Def The operator $\sim$ is a *equivalence relation on X* provided

$\sim$ is a relation on X and ⟨to get "equivalence" part⟩

(r) $\sim$ is *reflexive*, ie. $x \sim x$, $\forall x \in X$

(s) $\sim$ is *symmetric*, ie. $[x_1 \sim x_2] \Rightarrow [x_2 \sim x_1]$, $\forall x_1, x_2 \in X$

(t) $\sim$ is *transitive*, i.e. $[(x_1 \sim x_2) \text{ and } (x_2 \sim x_3)] \Rightarrow (x_1 \sim x_3)$, $\forall x_1, x_2, x_3 \in X$

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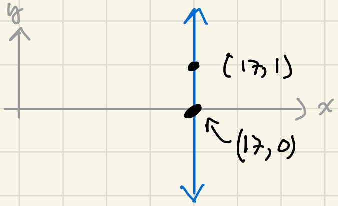
Ex 2 : ex. of equivalence relations

(1) $=$ on the set $\mathbb{R}$

(2) on $\mathbb{R}^2$ where $(x_1, y_1) \sim (x_2, y_2) \iff x_1 = x_2$.

So $\{(x,y) \in \mathbb{R}^2 : (x,y) \sim (17, y)\} := E_{(17,\underline{0})}$ = the line $x = 17$ ⟨(17,0) on on the line⟩

$\overset{\text{also}}{=} E_{(17,1)}$ ⟨ (17,1) on the line



(3) on the set X = the collection $\mathcal{S}$ of all sets where

$$S_1 \sim S_2 \overset{\text{def}}{\iff} \exists \text{ bij. } f : S_1 \to S_2 ,$$

Note. $\sim$ is a relation bcs either $\exists$ or $\nexists$ a bijection $f : S_1 \to S_2$.

Note $\sim$ is an equiv. relation bcs if $S_1, S_2, S_3, S \in \mathcal{S}$ then

(r) $S \sim S$, take the bij $f : S \to S$ given by $f(s) = s$. ⟨identity function⟩

(s) if $S_1 \sim S_2$, then $S_2 \sim S_1$ b/c

$\Downarrow$

if $\exists$ bij $f : S_1 \to S_2$ then $\underline{f^{-1}} : S_2 \to S_1$ is bij.

(t) Let $S_1 \sim S_2$ and $S_2 \sim S_3$. WTS $S_1 \sim S_3$.

$\Downarrow$ $\Downarrow$

$\exists$ bij $f : S_1 \to S_2$ $\exists$ bij $g : S_2 \to S_3$.

Then $\underline{g \circ f} : S_1 \to S_3$ is a bijection.

Def Let $\sim$ be a equiv. rel. on a set X,

For $x_0 \in X$, the <u>equivalence class E_{x_0} generated by x_0</u> is

$$E_{x_0} \overset{\text{def}}{=} \{x \in X : x \sim x_0\}$$

you can use either one for def.

Note

$$E_{x_0} = \{x \in X : x_0 \sim x\}$$

since $\sim$ is $\underline{\text{symmetric}}$ (so $x \sim x_0 \iff x_0 \sim x$).

Thm Two equiv classes are either disjnt or equal.

Rmk See Ex 2.2 ($\sim$ on $\mathbb{R}^2$ w/ equiv classes the vertical line)

Recall The equiv. class $E_{x_0} :\overset{\text{def}}{=} \{x \in X : x \sim x_0\}$.

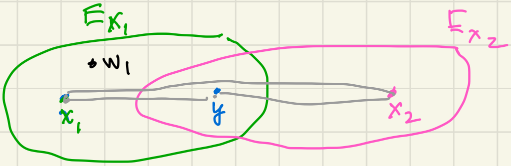
Pf

- Have an equiv rel. $\sim$ on a set X.
- Let $x_1, x_2 \in X$ s.t. $\underbrace{E_{x_1} \text{ and } E_{x_2} \text{ are not disjnt}}$. <WTS $E_{x_1} = E_{x_2}$>

$$\Downarrow$$

we can pick $y \in E_{x_1} \cap E_2$.

- Pic.



- note $x_1 \in E_{x_1}$ since $\sim$ is < $E_{x_1} \overset{\text{def}}{=} \{x : x \sim x_1\}$ and $\underset{\substack{\uparrow \\ \text{reflexive}}}{x_1 \sim x_1}$>

  Likewise $x_2 \in E_{x_2}$.

- **Claim 1** $x_1 \sim x_2$

$$\left[\begin{array}{l} \cdot\ y \in E_{x_1} \Rightarrow x_1 \sim y \\ \cdot\ y \in E_{x_2} \Rightarrow y \sim x_2 \end{array}\right] \xrightarrow[\text{transitive}]{\sim \text{ is}} x_1 \sim x_2$$

Claim 2 $E_{x_1} \subseteq E_{x_2}$

$$\left[\text{Let } w_1 \in E_{x_1} \Rightarrow \left.\begin{array}{l} w_1 \sim x_1 \\ x_1 \sim x_2 \end{array}\right] \xrightarrow[\text{transitive}]{\sim \text{ is}} w_1 \sim x_2 \overset{\text{def}}{\Rightarrow} w_1 \in E_{x_2} \right.$$

Know

E_{x_2}

Claim 3 $E_{x_2} \subseteq E_1$ is similar to Claim 2.

Claim 2 + Claim 3 $\Rightarrow E_{x_1} = E_{x_2}$ ☒

Ex/Def Let $B \sim \mathbb{N}$, So $\exists$ bij $f : \mathbb{N} \to B$

$$\begin{array}{ccc} 1 \bullet & \longrightarrow & \bullet\, b_1 \\ 2 \bullet & \longrightarrow & \bullet\, b_2 \\ 3 \bullet & \longrightarrow & \bullet\, b_3 \\ 4 \bullet & \longrightarrow & \bullet\, b_4 \\ \vdots & & \\ n \bullet & \longrightarrow & \bullet\, b_n \end{array}$$

the real math term

verb

aka enumerates

Key idea - The bij f "counts outs" all the elt's in B.

slang

Note

$$B = \biguplus_{n \in \mathbb{N}} \underbrace{\{b_n\}}_{\text{is a one point subset of } B} .$$

$\uplus$ is disjoint union

noun

Def Let $B \sim \mathbb{N}$. An enumeration of B is

$$B = \{b_1, b_2, b_3, b_4, \dots\}$$

where the function $f : \mathbb{N} \to B$ given by

$$f(n) = b_n$$

is a bijection.

Rmk So we can just say, eg,

Since $B \sim \mathbb{N}$, we can enumerate B as

$$B = \{b_1, b_2, b_3, \dots\}.$$

Why think this way: Bcs sometimes an enumeration is easier to work with than the bij $f : \mathbb{N} \to B$.

Recall $A \sim B \iff \exists$ bij btw A & B.

So "loosely speaking"

$A \sim B \iff$ "A & B" "have same # of elt".

It is ok to think (loosely) like "have same # of elt" when A & B have a finite # of elements but not ok when A and B are not finite (i.e. are infinite).

Thm 1.3.13 (Cantor). Let X be a set.
There does not exist a surjection $f : X \to \mathcal{P}(X)$ from the set X onto the power set $\mathcal{P}(X)$ of X.

Pf. LTGBG, <Let X be a set>.

- WLOG, X is nonempty since if $X = \emptyset$ then X has 0 elements but $\mathcal{P}(X) = \{\emptyset\}$ has 1 element so $\nexists$ surjection from X onto $\mathcal{P}(X)$.
- So let X be nonempty.
- BWOC, assume $\exists$ a surjection

$$f : \underset{\omega}{X} \to \underset{\omega}{\mathcal{P}(X)}. \qquad (*)$$

<WTF $\to\leftarrow$ >

< an elt in X → x $\quad$ B ← a subset of X
so mapping an elt to a set >

- Consider the subset D of X defined by

$$D = \{x \in X : \overset{\text{elt}}{x} \notin \overset{\text{set}}{f(x)}\} \qquad (1)$$

<So $x \in D \iff x$ maps to a set that does NOT contain x> ← brilliant idea of Cantor!

- Since $D \in \mathcal{P}(X)$ and $f : X \to \mathcal{P}(X)$ is surj., $\exists\, x_0 \in X$ st

$$f(x_0) = D. \qquad (2)$$

- **Claim 1** $x_0 \notin D$
[BWOC, assume $\underline{x_0 \in D}$. $\xRightarrow[(1)]{\text{def of } D} x_0 \notin f(x_0) \xRightarrow[(2)]{\text{def of } f} \underline{x_0 \notin D}$. $\to\leftarrow$

- **Claim 2** $x_0 \in D$
[BWOC, assume $\underline{x_0 \notin D}$. $\xRightarrow[(1)]{\text{def of } D} x_0 \in f(x_0) \xRightarrow[(2)]{\text{def of } f} \underline{x_0 \in D}$. $\to\leftarrow$

- Claim 1 and Claim 2 gives <note: Claim 1 says $x_0 \in D^c$>

$$x_0 \in D^c \cap D = \emptyset,$$

which is a contradiction.
Thus the original assumption in $(*)$ is false.
Thus $\nexists$ a surj. $f : X \xrightarrow{\text{onto}} \mathcal{P}(X)$. ∎

Cor If X is a set, then $X \nsim \mathcal{P}(X)$.