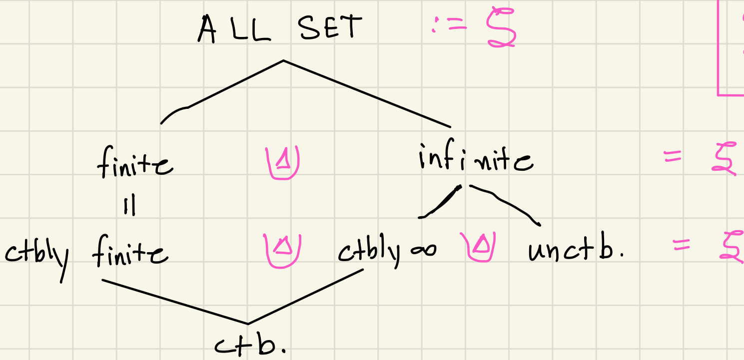
§ 1.3 Finite and Infinite Sets (aka Countability)

Overview Chart

$\uplus$ = disjoint union
ctb = countable
ctbly = countably
∞ = infinite



Rmk There are "levels of ∞". We will learn: $\mathbb{N}$ is ctbly infinite while $\mathbb{R}$ is unctb.

Def Sets A and B have the same cardinality (ie. cardinal number), written $A \sim B$, provided $\exists$ bijection between A and B.

Note $\sim$ is an equivalence relation <TL:rst>, ie: $A \sim A$ (reflexive), $[A \sim B \Rightarrow B \sim A]$ (symmetric), $[A \sim B$ and $B \sim C \Rightarrow A \sim C]$ (transitive).
Bcs f bij $\Rightarrow f^{-1}$ exists and f^{-1} bij. Bcs composite of 2 bij is bij.

Defs Let B be a set.

1. B is finite (aka ctbly finite) provided
   - B is nonempty and $\exists\, n \in \mathbb{N}$ st $B \sim \mathbb{N}^{\leq n}$ ← card $B = n$

   or

   - $B = \emptyset$ (and we write $B \sim \emptyset$) ← card $B = 0$.
2. B is ctbly infinite (aka enumerable, denumerable) provided $B \sim \mathbb{N}$.
3. B is countable provided B is ctbly finite or ctbly infinite.
4. B is uncountable provided B is not countable.
5. B is infinite provided B is not finite.

Warning. Some sources use the term countable to mean countably infinite.

Rmk. Since $\sim$ is transitive, B is ctbly $\infty \iff B \sim$ (some ctbly ∞ set)
and B is finite $\iff B \sim$ (some finite set).

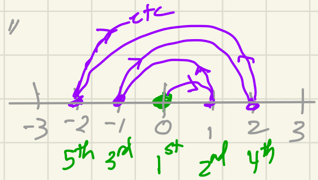
Rmk … 1.4.# is for "not in book" ∴ $\nexists$ § 1.4.

Prop 1.4.1 $\mathbb{N}$ is ctbly infinite, i.e. $\mathbb{N} \sim \mathbb{N}$.

Pf Follows from reflexivity of $\sim$.

Prop 1.4.2. $\mathbb{Z}$ is ctbly infinite, i.e. $\mathbb{Z} \sim \mathbb{N}$.

TL. How to "enumerate" i.e. list out, $\mathbb{Z}$.



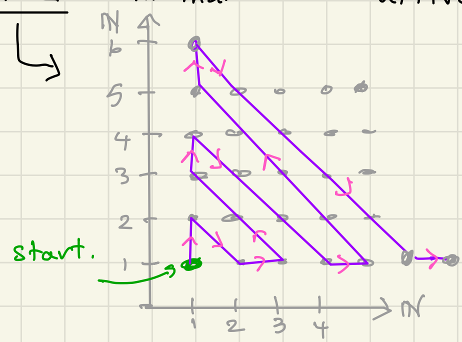
Pf. A bijection $f: \mathbb{N} \to \mathbb{Z}$ is

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ -\frac{(n-1)}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Prop 1.4.3 $\mathbb{N} \times \mathbb{N}$ is ctbly infinite, i.e. $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$.

Pf #1 To get bij. $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, let $f(m,k) = 2^{k-1}(2m-1)$ and use the fact that each $n \in \mathbb{N}$ has a unique factorization $n = 2^{k-1}(2m-1)$ with $m \in \mathbb{N}$ and $k \in \mathbb{N}$. This fact follows from the Prime Factorization Thm.

Pf #2 Informal but intuitive way to enumerate $\mathbb{N} \times \mathbb{N}$ = dots on grids
↳ "Cantor Snake"



Prop 1.4.4 TL diagram/pic.

$$\mathbb{N} \xrightarrow{f} \mathbb{N} \times \mathbb{N} \xrightarrow{\text{use } g_i\text{'s}} B_1 \times B_2$$

$$n \mapsto (f_1(n), f_2(n)) \mapsto (g_1(f_1(n)), g_2(f_2(n)))$$

Prop 1.4.4. Let $B_1 \sim \mathbb{N}$ and $B_2 \sim \mathbb{N}$. Then $B_1 \times B_2 \sim \mathbb{N}$.

Pf. LTGBG. Since $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$, $\exists$ bij. $f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, say $f(n) = (f_1(n), f_2(n))$.
For $i = 1$ or 2, since $B_i \sim \mathbb{N}$, $\exists$ bij $g_i: \mathbb{N} \to B_i$.
Define $h: \mathbb{N} \to B_1 \times B_2$ by $h(n) = (g_1(f_1(n)), \underbrace{g_2(\overbrace{f_2(n)}^{\in \mathbb{N}})}_{\in B_2})$.
Claim h is a bijection <for you>.

HW Read book's (short) proofs of Thms : 1.3.4, 1.3.5, 1.3.9.

Thm 1.3.4. Have sets : A, B, C.

(1a) $[A$ and B finite <u>and</u> disjnt$] \Rightarrow \operatorname{card}(A \uplus B) = \operatorname{card} A + \operatorname{card} B$.

(1b$^+$) $[A$ finite and $C \subseteq A] \Rightarrow \operatorname{card}(A \setminus C) = \operatorname{card} A - \operatorname{card} C$.

(1c) $[C$ infinite and B finite$] \Rightarrow C \setminus B$ is infinite.

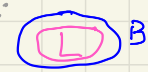
[pf by $\to\leftarrow$, use (1a)]

Rmks
- Book's (b) is for $\operatorname{card} C = 1$. Our (b$^+$) follow by induction using (b) for base step.
- For <u>finite</u> sets and cardinality, (a) and (b$^+$) roughly say "can do arithmetic".
- **!** For infinite sets and cardinality, "cannot do arithmetic" since there are "different levels of ∞". Luckily have (c) and below Thms.

Thm 1.3.5 / 1.3.9. Let $L \subseteq B$. Then

little $\leftarrow$ L , B $\rightarrow$ big

pic.



$\boxed{\text{By contrapositive: } a \Leftrightarrow b}$

(5a) B finite $\Rightarrow L$ finite

(5b) L infinite $\Rightarrow B$ infinite

(9a) B ctb $\Rightarrow L$ ctb

(9b) L unctb $\Rightarrow B$ unctb.

<u>Pf (9a)</u> for when B is ctbly infinite. LTGBG.

Enumerate $B = \{b_1, b_2, b_3, \dots\} = \{b_n : n \in \mathbb{N}\}$.

Construct a (strictly) increasing seq. $\{n_j\}_{j=1}^{\infty}$ of integers as follows

$\langle$ TL B: b_1 b_2 b_3 b_4 b_5 b_6 $\cdots$ $\leftarrow$ circle b_n's in L

L: b_{n_1} b_{n_2} b_{n_3} b_{n_4} $\rangle$

Let n_1 be the smallest $n \in \mathbb{N}$ st $b_n \in L$. So $b_{n_1} \in L$.

For $j \in \mathbb{N}$, having found $n_1, n_2, \dots, n_j$, let

let n_{j+1} be the smallest $n \in \mathbb{N}^{>n_j} \overset{\text{i.e.}}{=} \{1+n_j, 2+n_j, \dots\}$ st $b_n \in L$.

So $b_{n_{j+1}} \in L$ and $n_{j+1} > n_j$.

<u>Claim</u> L is enumerated by $\{b_{n_1}, b_{n_2}, b_{n_3}, \dots\} \overset{\text{i.e.}}{=} \{b_{n_j}\}_{j=1}^{\infty}$. $\langle$For you$\rangle$

Thm 1.4.5. Let $L \subseteq B$ and L be infinite and B be ctb.

TL: infinite $\uparrow$ $\uparrow$ ctb.

Then L is ctbly infinite, i.e. $L \sim \mathbb{N}$.

Pf. LTGBG. B ctb $\xRightarrow{\text{Thm 1.3.9a}}$ L is ctb.

$\xRightarrow{\text{given } L \text{ is infinite}}$ L is ctbly infinite.

Thm 1.3.12. A countable union of countable sets is countable. 13.1.4

Pf Let I be a countable index set.

Let $\{X_i\}_{i\in I}$ be a collection of countable sets.

We will show $\bigcup_{i\in I} X_i$ is a countable set.

WLOG, I is nonempty since $\bigcup_{i\in\emptyset} X_i = \emptyset$.

Step 1 Disjunctive the X_i's. I.e. find $\{D_i\}_{i\in I}$ s.t. $\bigcup_{i\in I} X_i = \biguplus_{i\in I} D_i$

set $D_1 = X_1$. For $k \in \mathbb{N}^{>1}$ let

$$D_k = X_k \setminus \bigcup_{j=1}^{k-1} X_j \overset{\text{i.e.}}{=} X_k \setminus (X_1 \cup X_2 \cup \dots \cup X_{k-1})$$

Step 2 Let $J = \{i \in I : D_i \text{ is nonempty}\} \overset{\text{note}}{\subseteq} I$. So $\biguplus_{j\in J} D_j = \biguplus_{i\in I} D_i = \bigcup_{i\in I} X_i$.

Step 3 For each $j \in J$, enumerate $D_j = \{d^j_n : n \in N_j\}$ where $N_j \subseteq \mathbb{N}$ is:

$$N_j = \begin{cases} \mathbb{N}^{\le \operatorname{card} D_j}, & \text{if } D_j \text{ is finite} \\ \mathbb{N}, & \text{if } D_j \text{ is infinite.} \end{cases}$$

Step 4 Define a map $f : \biguplus_{j\in J} D_j \to \mathbb{N}\times\mathbb{N}$ as follows.

If $x \in \biguplus_{j\in J} D_j$, then $\exists!\, j \in J$ st $x \in D_j$ and $\exists!\, n \in N_j$ s.t. $x = d^j_n$,

and so the map $f(x) \overset{\text{i.e.}}{=} f(d^j_n) = (j, n)$ is well-defined.

Since the D_j's are disjoint, f is injective.

Step 5 So $\overbrace{f[\bigcup_{j\in J} D_j]}^{\text{image of set}} \subseteq \overbrace{\mathbb{N}\times\mathbb{N}}^{\text{ctb}} \xRightarrow{\text{Thm 1.3.9a}} f[\bigcup_{j\in J} D_j]$ is ctb.

and $\underbrace{f[\bigcup_{j\in J} D_j]}_{\text{image of } \cup D_j} \underset{\text{b/c f is 1-to-1}}{\sim} \bigcup_{j\in J} D_j \underset{\text{sets are equal}}{\sim} \bigcup_{i\in I} X_i$

Thus $\bigcup_{i\in I} X_i$ is ctb.

Cor 1.4.6 A ctb. union of ctb. sets, where at least one of the sets is ctbly infinite, is ctbly infinite.

Pf Let $X_{i_0} \sim \mathbb{N}$. Then $\overbrace{X_{i_0}}^{\text{infinite}} \subseteq \overbrace{\bigcup_{i\in I} X_i}^{\text{ctb}}$. Thm 1.4.5 $\Rightarrow$ $\bigcup_{i\in I} X_i$ is ctb. inf.

Thm 1.3.11 $\mathbb{Q}$ is ctbly infinite, i.e. $\mathbb{Q} \sim \mathbb{N}$. 13.1.5

Pf. $\mathbb{Q} = \bigcup_{n \in \mathbb{N}} \{\frac{i}{n} : i \in \mathbb{Z}\}$ and $\{\frac{i}{n} : i \in \mathbb{Z}\} \sim \mathbb{Z} \sim \mathbb{N}$ $\Rightarrow$ $\mathbb{Q}$ is a ctb. union of ctbly ∞ sets $\xrightarrow[1.4.6]{\text{Cor}}$ $\mathbb{Q} \sim \mathbb{N}$.

Thm 1.4.7 The interval $(0,1)$ is uncountable. (see Thm 2.5.5)

Pf We showed in §2.5 (used Cantor diagonalization argument)

Thm There does not exist a bijection btw $(0,1)$ and $\mathbb{N}$.
So $(0,1)$ is not countably infinite. Also $(0,1)$ is not finite since
Also $(0,1)$ is not ctbly finite $(0,1) \supseteq \underbrace{\{\frac{1}{n} : n \in \mathbb{N}\}}_{\text{infinite}}$ by Thm 1.3.5 b.
So $(0,1)$ is not countable.

Thm 1.4.8. $\mathbb{R}$ is uncountable; infact, if $(0,1) \subseteq B$ the B is unctb.
Pf. $\underbrace{(0,1)}$ unctb. Now use Thm 1.3.9 b.

We close with a useful Thm to test for countability.

Thm 1.3.10 Let S be nonempty. TFAE.

(a) S is ctb. $\langle \exists$ bij. btw S and (either $\mathbb{N}$ or $\mathbb{N}_{\le n}$) $\rangle$

(b) $\exists$ surjection of $\mathbb{N}$ onto S $\langle \exists\, f : \mathbb{N} \xrightarrow{\text{onto}} S \rangle$

(c) $\exists$ injection of S into $\mathbb{N}$. $\langle \exists\, g : S \xrightarrow{\text{1-to-1}} \mathbb{N} \rangle$.

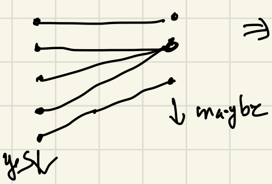
Pf and how to remember, clear: (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c).

(b) $\Rightarrow$ (a)

$\mathbb{N} \xrightarrow[\text{onto}]{f} S$

bcs f onto

$S = \bigcup_{n \in \mathbb{N}} \underbrace{f[\{n\}]}_{\text{ctb}}$

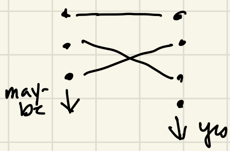


$\Rightarrow$

and ctb. union of ctb sets is ctb.

(c) $\Rightarrow$ (a)

$S \xrightarrow[1-1]{g} \mathbb{N}$



bcs g is 1-to-1

$S \sim g[S]$

and

$g[S] \subseteq \underbrace{\mathbb{N}}_{\text{ctb}}$

and a subset of ctb. set is ctb.

References

1. Bartle and Sherbert, Real Analysis 4^{th} ed, § 1.3 (p. 16 – 22).
2. Rudin, Principles of Mathematical Analysis, 3^{rd} ed, § 2.1 (p 24 – 30).