are not empty. (Why?) If the set A_x is bounded below, we set $a_x := \inf A_x$; if A_x is not bounded below, we set $a_x := -\infty$. Note that in either case $a_x \notin G$. If the set B_x is bounded above, we set $b_x := \sup B_x$; if B_x is not bounded above, we set $b_x := \infty$. Note that in either case $b_x \notin G$.

We now define $I_x := (a_x,\ b_x)$; clearly I_x is an open interval containing x. We claim that $I_x \subseteq G$. To see this, let $y \in I_x$ and suppose that $y < x$. It follows from the definition of a_x that there exists $a' \in A_x$ with $a' < y$, whence $y \in (a', x] \subseteq G$. Similarly, if $y \in I_x$ and $x < y$, there exists $b' \in B_x$ with $y < b'$, whence it follows that $y \in [x,\ b') \subseteq G$. Since $y \in I_x$ is arbitrary, we have that $I_x \subseteq G$.

Since $x \in G$ is arbitrary, we conclude that $\bigcup_{x \in G} I_x \subseteq G$. On the other hand, since for each $x \in G$ there is an open interval I_x with $x \in I_x \subseteq G$, we also have $G \subseteq \bigcup_{x \in G} I_x$. Therefore we conclude that $G = \bigcup_{x \in G} I_x$.

We claim that if $x,\ y \in G$ and $x \neq y$, then either $I_x = I_y$ or $I_x \cap I_y = \emptyset$. To prove this suppose that $z \in I_x \cap I_y$, whence it follows that $a_x < z < b_y$ and $a_y < z < b_x$. (Why?) We will show that $a_x = a_y$. If not, it follows from the Trichotomy Property that either (i) $a_x < a_y$, or (ii) $a_y < a_x$. In case (i), then $a_y \in I_x = (a_x, b_x) \subseteq G$, which contradicts the fact that $a_y \notin G$. Similarly, in case (ii), then $a_x \in I_y = \left(a_y, b_y\right) \subseteq G$, which contradicts the fact that $a_x \notin G$. Therefore we must have $a_x = a_y$ and a similar argument implies that $b_x = b_y$. Therefore, we conclude that if $I_x \cap I_y \neq \emptyset$, then $I_x = I_y$.

It remains to show that the collection of distinct intervals $\{I_x : x \in G\}$ is countable. To do this, we enumerate the set $\mathbb{Q}$ of rational numbers $\mathbb{Q} = \{r_1, r_2, \ldots, r_n, \ldots\}$ (see Theorem 1.3.11). It follows from the Density Theorem 2.4.8 that each interval I_x contains rational numbers; we select the rational number in I_x that has the smallest index n in this enumeration of $\mathbb{Q}$. That is, we choose $r_{n(x)} \in \mathbb{Q}$ such that $I_{r_{n(x)}} = I_x$ and $n(x)$ is the smallest index n such that $I_{r_n} = I_x$. Thus the set of distinct intervals $I_x,\ x \in G$, is put into correspondence with a subset of $\mathbb{N}$. Hence this set of distinct intervals is countable. Q.E.D.

It is left as an exercise to show that the representation of G as a disjoint union of open intervals is uniquely determined.

It does *not* follow from the preceding theorem that a subset of $\mathbb{R}$ is closed if and only if it is the intersection of a countable collection of closed *intervals* (why not?). In fact, there are closed sets in $\mathbb{R}$ that cannot be expressed as the intersection of a countable collection of closed intervals in $\mathbb{R}$. A set consisting of two points is one example. (Why?) We will now describe the construction of a much more interesting example called the Cantor set.

The Cantor Set

The Cantor set, which we will denote by $\mathbb{F}$, is a very interesting example of a (somewhat complicated) set that is unlike any set we have seen up to this point. It reveals how inadequate our intuition can sometimes be in trying to picture subsets of $\mathbb{R}$.

The Cantor set $\mathbb{F}$ can be described by removing a sequence of open intervals from the closed unit interval $I := [0,\ 1]$. We first remove the open middle third $\left(\frac{1}{3},\ \frac{2}{3}\right)$ of $[0, 1]$ to obtain the set

$$F_1 := \left[0,\ \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, 1\right].$$

We next remove the open middle third of each of the two closed intervals in F_1 to obtain the set

$$F_2 := \left[0,\ \tfrac{1}{9}\right] \cup \left[\tfrac{2}{9},\ \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3},\ \tfrac{7}{9}\right] \cup \left[\tfrac{8}{9},\ 1\right].$$

We see that F_2 is the union of $2^2 = 4$ closed intervals, each of which is of the form $\left[k/3^2, (k+1)/3^2\right]$. We next remove the open middle thirds of each of these sets to get F_3, which is the union of $2^3 = 8$ closed intervals. We continue in this way. In general, if F_n has been constructed and consists of the union of 2^n intervals of the form $[k/3^n, (k+1)/3^n]$, then we obtain the set F_{n+1} by removing the open middle third of each of these intervals. The Cantor set $\mathbb{F}$ is what remains after this process has been carried out for every $n \in \mathbb{N}$. (See Figure 11.1.1.)

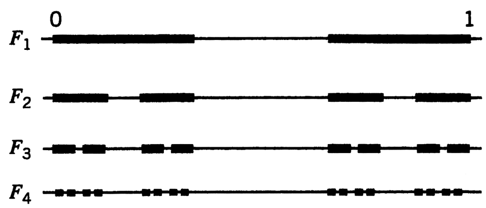

Figure 11.1.1 Construction of the Cantor set

11.1.10 Definition The **Cantor set** $\mathbb{F}$ is the intersection of the sets F_n, $n \in \mathbb{N}$, obtained by successive removal of open middle thirds, starting with [0, 1].

Since it is the intersection of closed sets, $\mathbb{F}$ is itself a closed set by 11.1.5(a). We now list some of the properties of $\mathbb{F}$ that make it such an interesting set.

(1) The total length of the removed intervals is 1.

We note that the first middle third has length $1/3$, the next two middle thirds have lengths that add up to $2/3^2$, the next four middle thirds have lengths that add up to $2^2/3^3$, and so on. The total length L of the removed intervals is given by

$$L = \frac{1}{3} + \frac{2}{3^2} + \cdots + \frac{2^n}{3^{n+1}} + \cdots = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n.$$

Using the formula for the sum of a geometric series, we obtain

$$L = \frac{1}{3} \cdot \frac{1}{1-(2/3)} = 1.$$

Thus $\mathbb{F}$ is a subset of the unit interval [0, 1] whose complement in [0, 1] has total length 1.

Note also that the total length of the intervals that make up F_n is $(2/3)^n$, which has limit 0 as $n \to \infty$. Since $\mathbb{F} \subseteq F_n$ for all $n \in \mathbb{N}$, we see that *if* $\mathbb{F}$ can be said to have "length," it must have length 0.

(2) The set $\mathbb{F}$ contains no nonempty open interval as a subset.

Indeed, if $\mathbb{F}$ contains a nonempty open interval $J := (a, b)$, then since $J \subseteq F_n$ for all $n \in \mathbb{N}$, we must have $0 < b - a \le (2/3)^n$ for all $n \in \mathbb{N}$. Therefore $b - a = 0$, whence J is empty, a contradiction.

(3) The Cantor set $\mathbb{F}$ has infinitely (even uncountably) many points.

The Cantor set contains all of the endpoints of the removed open intervals, and these are all points of the form $2^k/3^n$ where $k = 0, 1, \ldots, n$ for each $n \in \mathbb{N}$. There are infinitely many points of this form.

The Cantor set actually contains many more points than those of the form $2^k/3^n$; in fact, $\mathbb{F}$ is an uncountable set. We give an outline of the argument. We note that each

$x \in [0, 1]$ can be written in a ternary (base 3) expansion

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} = (.a_1 a_2 \cdots a_n \cdots)_3$$

where each a_n is either 0 or 1 or 2. (See the discussion at the end of Section 2.5.) Indeed, each x that lies in one of the removed open intervals has $a_n = 1$ for some n; for example, each point in $\left(\frac{1}{3}, \frac{2}{3}\right)$ has $a_1 = 1$. The endpoints of the removed intervals have two possible ternary expansions, one having no 1s; for example, $3 = (.100\cdots)_3 = (.022\cdots)_3$. If we choose the expansion without 1s for these points, then $\mathbb{F}$ consists of all $x \in [0, 1]$ that have ternary expansions with no 1s; that is, a_n is 0 or 2 for all $n \in \mathbb{N}$. We now define a mapping φ of $\mathbb{F}$ onto [0, 1] as follows:

$$\varphi\left(\sum_{n=1}^{\infty} \frac{a_n}{3^n}\right) := \sum_{n=1}^{\infty} \frac{(a_n/2)}{2^n} \qquad \text{for} \quad x \in \mathbb{F}.$$

That is, $\varphi\big((.a_1 a_2 \cdots)_3\big) = (.b_1 b_2 \cdots)_2$ where $b_n = a_n/2$ for all $n \in \mathbb{N}$ and $(.b_1 b_2 \cdots)_2$ denotes the *binary* representation of a number. Thus φ is a surjection of $\mathbb{F}$ onto [0, 1]. Assuming that $\mathbb{F}$ is countable, Theorem 1.3.10 implies that there exists a surjection ψ of $\mathbb{N}$ onto $\mathbb{F}$, so that $\varphi \circ \psi$ is a surjection of $\mathbb{N}$ onto [0, 1]. Another application of Theorem 1.3.10 implies that [0, 1] is a countable set, which contradicts Theorem 2.5.5. Therefore $\mathbb{F}$ is an uncountable set.

Exercises for Section 11.1

1. If $x \in (0, 1)$, let ε_x be as in Example 11.1.3(b). Show that if $|u - x| < \varepsilon_x$, then $u \in (0, 1)$.
2. Show that the intervals (a, ∞) and $(-\infty, a)$ are open sets, and that the intervals $[b, \infty)$ and $(-\infty, b]$ are closed sets.
3. Write out the Induction argument in the proof of part (b) of the Open Set Properties 11.1.4.
4. Prove that $(0, 1] = \bigcap_{n=1}^{\infty} (0, 1 + 1/n)$, as asserted in Example 11.1.6(a).
5. Show that the set $\mathbb{N}$ of natural numbers is a closed set in $\mathbb{R}$.
6. Show that $A = \{1/n : n \in \mathbb{N}\}$ is not a closed set, but that $A \cup \{0\}$ is a closed set.
7. Show that the set $\mathbb{Q}$ of rational numbers is neither open nor closed.
8. Show that if G is an open set and F is a closed set, then $G \backslash F$ is an open set and $F \backslash G$ is a closed set.
9. A point $x \in \mathbb{R}$ is said to be an **interior point** of $A \subseteq \mathbb{R}$ in case there is a neighborhood V of x such that $V \subseteq A$. Show that a set $A \subseteq \mathbb{R}$ is open if and only if every point of A is an interior point of A.
10. A point $x \in \mathbb{R}$ is said to be a **boundary point** of $A \subseteq \mathbb{R}$ in case every neighborhood V of x contains points in A and points in $\mathcal{C}(A)$. Show that a set A and its complement $\mathcal{C}(A)$ have exactly the same boundary points.
11. Show that a set $G \subseteq \mathbb{R}$ is open if and only if it does not contain any of its boundary points.
12. Show that a set $F \subseteq \mathbb{R}$ is closed if and only if it contains all of its boundary points.
13. If $A \subseteq \mathbb{R}$, let A° be the union of all open sets that are contained in A; the set A° is called the **interior** of A. Show that A° is an open set, that it is the largest open set contained in A, and that a point z belongs to A° if and only if z is an interior point of A.

14. Using the notation of the preceding exercise, let A, B be sets in $\mathbb{R}$. Show that $A^\circ \subseteq A$, $(A^\circ)^\circ = A^\circ$, and that $(A \cap B)^\circ = A^\circ \cap B^\circ$. Show also that $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$, and give an example to show that the inclusion may be proper.
15. If $A \subseteq \mathbb{R}$, let A^- be the intersection of all closed sets containing A; the set A^- is called the **closure** of A. Show that A^- is a closed set, that it is the smallest closed set containing A, and that a point w belongs to A^- if and only if w is either an interior point or a boundary point of A.
16. Using the notation of the preceding exercise, let A, B be sets in $\mathbb{R}$. Show that we have $A \subseteq A^-$, $(A^-)^- = A^-$, and that $(A \cup B)^- = A^- \cup B^-$. Show that $(A \cap B)^- \subseteq A^- \cap B^-$, and give an example to show that the inclusion may be proper.
17. Give an example of a set $A \subseteq \mathbb{R}$ such that $A^\circ = \emptyset$ and $A^- = \mathbb{R}$.
18. Show that if $F \subseteq \mathbb{R}$ is a closed nonempty set that is bounded above, then $\sup F$ belongs to F.
19. If G is open and $x \in G$, show that the sets A_x and B_x in the proof of Theorem 11.1.9 are not empty.
20. If the set A_x in the proof of Theorem 11.1.9 is bounded below, show that $a_x := \inf A_x$ does not belong to G.
21. If in the notation used in the proof of Theorem 11.1.9, we have $a_x < y < x$, show that $y \in G$.
22. If in the notation used in the proof of Theorem 11.1.9, we have $I_x \cap I_y \neq \emptyset$, show that $b_x = b_y$.
23. Show that each point of the Cantor set $\mathbb{F}$ is a cluster point of $\mathbb{F}$.
24. Show that each point of the Cantor set $\mathbb{F}$ is a cluster point of $\mathcal{C}(\mathbb{F})$.

Section 11.2 Compact Sets

In advanced analysis and topology, the notion of a ''compact'' set is of enormous importance. This is less true in $\mathbb{R}$ because the Heine-Borel Theorem gives a very simple characterization of compact sets in $\mathbb{R}$. Nevertheless, the definition and the techniques used in connection with compactness are very important, and the real line provides an appropriate place to see the idea of compactness for the first time.

The definition of compactness uses the notion of an open cover, which we now define.

11.2.1 Definition Let A be a subset of $\mathbb{R}$. An **open cover** of A is a collection $\mathcal{G} = \{G_\alpha\}$ of open sets in $\mathbb{R}$ whose union contains A; that is,

$$A \subseteq \bigcup_\alpha G_\alpha.$$

If $\mathcal{G}'$ is a subcollection of sets from $\mathcal{G}$ such that the union of the sets in $\mathcal{G}'$ also contains A, then $\mathcal{G}'$ is called a **subcover** of $\mathcal{G}$. If $\mathcal{G}'$ consists of finitely many sets, then we call $\mathcal{G}'$ a **finite subcover** of $\mathcal{G}$.

There can be many different open covers for a given set. For example, if $A := [1, \infty)$, then the reader can verify that the following collections of sets are all open covers of A:

$$\begin{aligned}
\mathcal{G}_0 &:= \{(0, \infty)\}, \\
\mathcal{G}_1 &:= \{(r-1,\ r+1) : r \in \mathbb{Q}, r > 0\}, \\
\mathcal{G}_2 &:= \{(n-1,\ n+1) : n \in \mathbb{N}\}, \\
\mathcal{G}_3 &:= \{(0,\ n) : n \in \mathbb{N}\}, \\
\mathcal{G}_4 &:= \{(0,\ n) : n \in \mathbb{N}, n \geq 23\}.
\end{aligned}$$