Math 554      Fall 2007      Final

### Mark Box

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**Instruction**

1. As in the past... all the same!! - including no books/notes.
2. no calculators.
3. if need more pages - just ask.
1-1 (All of) the subsequential limit points of \( IT \cup \{ \pm \infty \} \) of 
the sequence \( \{ (-1)^n \} \) is/are: \( \{ 1, -1 \} \)

1-2 (All of) the subsequential limit points of \( IT \cup \{ \pm \infty \} \) of 
the sequence \( \{ \frac{1}{n} \cos \frac{n\pi}{3} \} \) is/are: \( \{ -\infty, 0, +\infty \} \)

1-3 The Bolzano-Weierstrass Theorem says that each bounded subset of \( IT \) has a limit point.

1-4 According to the BW theorem is that each bounded sequence in \( IT \) has a convergent subsequence.

1-5 Define a sequence \( \{ y_n \} \) by \( y_1 = 1 \) and \( y_n := \frac{1}{4} (2y_{n-1} + 3) \) for \( n \geq 2 \).

With some work one can show that \( \{ y_n \} \) is a monotone increasing sequence that is bounded above by \( 2 \). Once we know this, we can show that \( \lim_{n \to \infty} y_n = \frac{3}{2} \).

\( L = \frac{1}{4} (2L + 3) \Rightarrow 4L = 2L + 3 \Rightarrow 2L = 3 \)

1-6 \( \lim_{n \to \infty} \frac{\cos \left( \frac{n\pi}{3} \right)}{n^{17}} = 0 \) \quad \text{since} \quad \cos(n\pi) = (-1)^n \quad \text{for} \quad n \text{ even,} \quad \frac{1}{3} \text{ for } n \text{ odd}

1-7 \( \lim_{n \to \infty} \frac{2^n}{3^n - 1} = 0 \)
1-8 \( \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) = \emptyset \)

1-9 \( \bigcap_{n=1}^{\infty} \left[ 0, 1-\frac{1}{n} \right] = \emptyset \)

Thus 3: all are subsets of \( \mathbb{R} \).

1-10 \( \bigcap_{n=1}^{\infty} \left[ n, \infty \right) = \emptyset \)

TF (1-11) \( C \setminus (A \cap B) = (C \setminus A) \cap (C \setminus B) \) de Morgan

\( \Rightarrow \) A and B are subsets of C.

TF (1-12) A function \( f : X \rightarrow Y \) is 1-to-1 \( \iff \)

\( \text{if } f(x_1) \neq f(x_2) \text{ then } x_1 \neq x_2 \).

TF (1-13) If a function \( f : \mathbb{N} \rightarrow A \) is onto, then \( A \) is at most countable.

TF (1-14) If \( A \) is countable and \( \emptyset \neq B \) and \( B \) is finite, then \( A \times B \) is countable.

TF (1-15) If \( X \) and \( Y \) are nonempty sets and \( X \times Y \), then \( \mathcal{P}(X) \sim \mathcal{P}(Y) \).

[see first page of problem 5 for helpful diagram and notation explanation.]
(1-16) A convergent sequence in $\mathbb{R}$ is bounded.

(1-17) A bounded sequence in $\mathbb{R}$ is convergent.

(1-18) Let $\{a_n\}$ and $\{b_n\}$ be seq. in $\mathbb{R}$. If $\{a_n\}$ converges and $\{b_n\}$ is bounded, then $\{a_n + b_n\}$ converges.

(1-19) Let $\{a_n\}$ and $\{b_n\}$ be seq. in $\mathbb{R}$. If $\{a_n\}$ converges to zero and $\{b_n\}$ is bounded, then $\{a_n \cdot b_n\}$ converges to zero.

(1-20) Prof. Girardi wishes the class a peaceful holiday and a healthy New Year.

Of course, this is true but Prof. Girardi will accept any answer. 😊
2 Sets and Functions

Let $X$ and $Y$ be sets and $f$ be a function from $X$ to $Y$, so
\[ f : X \to Y. \]

Let $A \subseteq X$ and $B \subseteq Y$.

Let $I$ be an indexing set and $A_\beta \subseteq X$ for each $\beta \in I$.

(2a) Fill-in blanks.

(2a-1) \( x \in \bigcap_{\beta \in I} A_\beta \iff \forall \beta \in I, x \in A_\beta. \)

(2a-2) By definition of (direct) image, \( y \in f(A) \iff \exists x \in A \text{ such that } y = f(x). \)

(2a-3) By definition of inverse (pre-)image, \( x \in f^{-1}(B) \iff f(x) \in B. \)

(2a-4) The inverse function \( f^{-1} : Y \to X \) of $f$ exists $\iff f$ is bijective.

(2b) Prove one, but not both, of (2b-1) and (2b-2). Space is provided next 2 pages.

(2b-1) Prove that \((f \circ f^{-1})(B) \subseteq B.\)

Also, give an example showing that set equality need not hold.

(2b-2) Prove that \( f \left( \bigcap_{\beta \in I} A_\beta \right) \subseteq \bigcap_{\beta \in I} f(A_\beta). \)

Also, give an example showing that set equality need not hold.

Hints:
- For examples, a picture with everything clearly labeled and a brief explanation is enough.
- For the example in (2b-2), it is enough to do in the special case $I = \{1, 2, 3\}$ and find an example where \( f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2). \)
(2c) Worth 2 pts -

(2c-1) I am doing *(2b-1) or *(2b-2)*. (Circle one)

(2c-2) Check one box (1 pt for correct answer).

☑ I believe my proof is totally (or at least very close to being) correct.
☐ I believe my proof is a good start towards a correct proof but there are some mistakes/holes.
☐ I am just rambling in hope of partial credit.
☐ Other. Explain:

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Space for (2b)

2b-1) \text{wts.} \quad (f\circ f^{-1})(B) \subseteq B

Proof. Let \( y \in (f \circ f^{-1})(B) = f(f^{-1}(B)) \), then, by definition of image,

\[ \exists x \in f^{-1}(B) \text{ such that } f(x) = y. \]

But then, by definition of inverse image, \( x \in f^{-1}(B) \implies f(x) \in B \).

Since \( f(x) = y, \quad y \in B \).

Thus, \( (f \circ f^{-1})(B) \subseteq B \) since \( y \in (f \circ f^{-1})(B) \implies y \in B \quad \forall y \in (f \circ f^{-1})(B) \).

Example. \[ \text{Let } A = \{ 0, 1 \} \text{ and } B = \{ 0, 13 \}. \text{ Let } f: A \rightarrow B \]

such that \( f(1) = 1 \). Then \( (f \circ f^{-1})(B) = \{ 1 \} \subseteq \{ 0, 13 \} = B \).
(2c) Worth 2 pts -

I am doing (2b-1) or (2b-2).  (circle one)

Check one box (1 pt for correct answer).

☐ I believe my proof is totally (or at least very close to being) correct.

☐ I believe my proof is a good start towards a correct proof, but there are some mistakes/holes.

☐ I am just rambling in hope of partial credit.

☐ Other.  Explain:

Space for (2b) → (2b-2)

Proof

Let the givens be given. We will show that

\[ f(\bigcap_{\beta \in I} A_\beta) = \bigcap_{\beta \in I} f(A_\beta) \quad (\#) \]

Fix \( y \in f(\bigcap_{\beta \in I} A_\beta) \). By definition of the image of a set,

there exists \( x \in \bigcap_{\beta \in I} A_\beta \) such that \( y = f(x) \). Thus, for each \( \beta \in I \),

\[ x \in A_\beta \]

by def of intersection and

\[ f(x) \in f(A_\beta) \]

by def of image of a set and

\[ y \in f(A_\beta) \]

since \( y = f(x) \).

So by definition of intersection, \( y \in \bigcap_{\beta \in I} f(A_\beta) \).

So (\#) holds.
#3 Least Upper Bound (lub) and Greatest Lower Bound (glb)

Let \( \emptyset \neq A \subseteq \mathbb{R} \) and \( A \) be bounded above.

Let \( \alpha \in \mathbb{R} \).

\[
\begin{align*}
\text{or (ii') } \cdots \text{ then } & \exists a \in A \text{ s.t. } \beta < a \leq \alpha \text{.} \\
\text{or (ii'') } \cdots \text{ then } & \exists a \in A \text{ s.t. } \beta < a \text{.}
\end{align*}
\]

(3a) Fill in the blanks.

\( \alpha = \text{lub } A \iff \text{(i) } \alpha \text{ is an upper bound of } A \)

\( \text{(ii) if } \beta \in \mathbb{R} \text{ and } \beta < \alpha, \text{ then } \beta \text{ is not an upper bound of } A \)

(3a-2) If \( \alpha = \text{lub } A \), is it always true that \( \alpha \in A \)? NO (Yes/No)

(3a-3) State the least upper bound property of \( \mathbb{R} \).

Each nonempty subset of \( \mathbb{R} \) that is bounded above has a supremum in \( \mathbb{R} \).

\( \text{i.e. If } \emptyset \neq A \subseteq \mathbb{R} \text{ and } A \text{ is bounded above, then } \sup A \text{ exists and is in } \mathbb{R} \).

(3b) Prove one, but not both, of (3b-1) and (3b-2). Space provided next 2 pages.

(3b-1) Let \( b \in \mathbb{R} \) and define the set \( b + A \) by

\[
b + A := \{ b + a \in \mathbb{R} : a \in A \}.
\]

Prove that

\[
\text{lub } (b + A) = b + \text{lub } A.
\]

(You may use, without proving, that \( b + A \) is bounded above since \( A \) is bounded above)

(3b-2) Let \( \alpha = \text{lub } A \) and \( \alpha \notin A \). Prove that for each \( \epsilon > 0 \), the interval \( (\alpha - \epsilon, \alpha) \) contains infinitely many points of \( A \).
(3c) Worth 2 pts.

(3c-1) I am doing (3b-1) or (3b-2). (circle one)

(3c-2) Check one box (1 pt for correct answer).

☐ I believe my proof is totally (or at least very close to being) correct.
☐ I believe my proof is a good start towards a correct proof but there are some mistakes/holes.
☐ I am just rambling in hopes of partial credit.
☐ Other. Explain:

Space for (3b)

Let the givens be given and define

$$b + A := \exists b + a \in \mathbb{R} : a \in A.$$  

(WTS \(1_{\text{ub}}(b+A) = b + 1_{\text{ub}} A\).)

Define \(2 := 1_{\text{ub}} A\) and let \(\beta = 1_{\text{ub}}(b+A)\).

We will show inequality in both directions to prove equality.

[\text{3}] Take \(\beta = 1_{\text{ub}}(b+A)\).

Then \(\beta = b + a \forall a \in \mathbb{R} + A\), by definition of \(1_{\text{ub}}\).

So, \(\beta - b = a \forall a \in A\) by simple algebra.

(Continue on next page →)
So, $\beta - b$ is an upper bound of $A$.

So, $\beta - b \geq \alpha$ since $\alpha$ is the least upper bound of $A$.

Finally, $\beta \geq \alpha + b$.

Let $a = \text{lub}(A)$.

So, $a \geq \alpha$ as $A$ by definition of lub.

Therefore, $\alpha + b \geq a + b$ and $a + b \in A$.

So, $\beta = \alpha + b$ is an upper bound of $b + A$.

So, $\alpha + b \geq \beta$ since $\beta$ is the least upper bound of $b + A$.

By showing inequality both ways,

\[ \text{lub}(b + A) = b + \text{lub}(A). \]

\[ \text{QED} \]
(3c) Worth 2 pts.

(3c-1) I am doing (3b-1) or (3b-2). (circle one)

(3c-2) Check one box. (1 pt for correct answer).

☑ I believe my proof is totally (or at least very close to being) correct.
☐ I believe my proof is a good start towards a correct proof but there are some mistakes/holes.
☐ I am just rambling in hopes of partial credit.
☐ Other. Explain:

Space for (3b)

**Theorem:** If \( A \) is a non-empty subset of the reals and bounded above with \( \alpha = \text{ lub}(A) \) and \( \alpha \notin A \), then \( \forall \varepsilon > 0 \) the interval \((\alpha - \varepsilon, \alpha)\) contains infinitely many points of \( A \).

**Proof:** Fix some \( \varepsilon > 0 \). Since \( \varepsilon > 0 \), \( \alpha - \varepsilon < \alpha \) and hence \( \alpha - \varepsilon \) is not an upper bound of \( A \). Thus \( \exists a \in A \) s.t. \( \alpha - \varepsilon < a \leq \alpha \). Since \( a \notin A \) while \( a \in A \), \( a \neq a \) and hence \( a < \alpha \).

Thus \( \alpha - \varepsilon < a < \alpha \). Since \( a \in (\alpha - \varepsilon, \alpha) \), the set \((\alpha - \varepsilon, \alpha) \cap A\) is non-empty. \((\alpha - \varepsilon, \alpha) \cap A\) can be either finite or infinite. We will prove it is infinite by contradiction.

Assume there are only a finite number of points in \((\alpha - \varepsilon, \alpha) \cap A\). Since \((\alpha - \varepsilon, \alpha) \cap A\) is finite and non-empty, we can pick the largest element and call it \( a_0 \). \( a_0 \in (\alpha - \varepsilon, \alpha) \) implies \( a_0 < \alpha \) and hence by the definition of \( \text{ lub}(A) \), \( \exists \alpha \in A \) s.t. \( a_0 < a < \alpha \). However, \( a \notin A \) implies then that \( a_0 < a < \alpha \). We see then that \( a \in (\alpha - \varepsilon, \alpha) \cap A \) and \( a \neq a_0 \). This however, = Continued next page
is a contradiction as we chose \( a_0 \) to be the largest member of the finite set \((\alpha - \epsilon, \alpha) \cap A\). Thus \((\alpha - \epsilon, \alpha) \cap A\) must be infinite, which is to say the interval \((\alpha - \epsilon, \alpha)\) contains infinitely many points of \(A\).

Way #2

Let \( \emptyset \neq A \subseteq \mathbb{R} \) and \( A \) be bounded above. Thus \( A \) has a least upper bound \( \alpha \). Let \( \alpha = \text{lub} A \). Also let \( \alpha \notin A \). Fix \( \epsilon > 0 \).

We will show that the interval \((\alpha - \epsilon, \alpha)\) has infinitely many points of \(A\).

Since \( \alpha = \text{lub} A \), \( \alpha - \epsilon \) is not an upper bound of \( A \).
So \( \exists \ a_1 \in A \) s.t. \( \alpha - \epsilon < a_1 \leq \alpha \). But \( \alpha \notin A \) and \( a_1 \in A \).
So \( \alpha - \epsilon < a_1 < \alpha \).

Since \( \alpha = \text{lub} A \) and \( a_1 < \alpha \), \( a_1 \) is not an upper bound of \( A \).
So \( \exists \ a_2 \in A \) s.t. \( a_1 < a_2 \leq \alpha \). But \( \alpha \notin A \) and \( a_2 \in A \).
So \( a_1 < a_2 < \alpha \).

Continue in this fashion to obtain a set \( S = \{ a_n : n \in \mathbb{N} \} \) such that \( a_n \in A \) for each \( n \in \mathbb{N} \) and
\[
\alpha - \epsilon < a_1 < a_2 < a_3 < a_4 < \ldots < \alpha.
\]

Clearly \( S \subseteq A \cap (\alpha - \epsilon, \alpha) \). Since the \( a_n \)'s are strictly increasing, the \( a_n \)'s are distinct so \( S \) is infinite.
4. Sequences

Let \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) be sequences in \( \mathbb{R} \).

Let \( a \in \mathbb{R} \).

(4a) Fill in blanks.

(4a-1) By definition, \( \{a_n\} \) converges to \( a \) (also expressed as \( \lim_{n \to \infty} a_n = a \)) if and only if \( \forall \varepsilon > 0 \) \( \exists K \in \mathbb{N} \) such that \( (n \in \mathbb{N} \land n \geq K) \Rightarrow a_n \in N\varepsilon(a) \).

(4a-2) By definition, \( \{a_n\} \) diverges if \( \{a_n\} \) does not converge.

(4a-3) By definition, \( \{a_n\} \) converges to \( +\infty \) if \( \forall M \in \mathbb{R} \) \( \exists N \in \mathbb{N} \) such that \( (n \in \mathbb{N} \land n \geq N) \Rightarrow a_n > M \).

(4a-4) A monotone decreasing sequence \( \{a_n\} \) converges to a (finite) real number if and only if \( \{a_n\} \) is bounded below.

(4b) Prove one, but not both, of (4b-1) and (4b-2). Space is provided next 2 pages.

(4b-1) Let \( \{a_n\} \) converge to \( a \). Let \( \{b_n\} \) satisfy that \( \forall \varepsilon > 0 \) \( \exists M, N \in \mathbb{N} \) such that \( (n \geq M \land n \geq N) \Rightarrow |a_n - b_n| < \varepsilon \). Show that \( \{b_n\} \) converges and find the limit \( \lim_{n \to \infty} b_n \).

Remark: Figure out what \( \{b_n\} \) converges to and then use an \( \varepsilon-N \) argument to show that \( \{b_n\} \) does converge to that number.

(4b-2) Let \( \emptyset \neq A \subseteq \mathbb{R} \) and \( A \) be bounded above. Let \( \alpha := \text{lub} A \in \mathbb{R} \).

Show that (i.e., explain how to construct) there is a sequence \( \{a_n\} \) such that \( a_n \in A \) for each \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} a_n = \alpha \).

Hint: Case 1: Let \( \alpha \in A \). Case 2: Let \( \alpha \notin A \).

Case 1 is easy. For Case 2, you may use without proving (4b-2) of this exam.
(4c) Worth 2 pts.

(4c-1) I am doing (4b-1) or (4b-2). (circle one).

(4c-2) Check one box (1 pt for correct answer).

☐ I believe my proof is totally (or at least very close to being) correct.

☐ I believe my proof is a good start towards a correct proof but there are some mistakes/holes.

☐ I am just rambling in hopes of partial credit.

☐ Other. Explain:

Space for (4b)

\[ 4b-1 \quad a_n \to a, (\forall \varepsilon > 0) \exists N \in \mathbb{N} \left[ n > N \implies |a_n - bn| < \frac{\varepsilon}{3} \right] \]

Proof. Fix \( \varepsilon > 0 \). Let \( N \in \mathbb{N} \) st. \( \forall n \in \mathbb{N}, n \geq N, \quad |a_n - bn| < \frac{\varepsilon}{3} \).

Existence: since we can choose any \( \varepsilon, \frac{\varepsilon}{3} \), and find an \( M \) big enough so that \( |a_n - bn| < \frac{\varepsilon}{3} \) for all \( n > M \). Now, since \( a_n \to a \), \( \exists N_0 \in \mathbb{N} \) such that

\[ |a_n - a| < \frac{\varepsilon}{3} \quad \text{for} \quad n > N_0. \]

By the triangle inequality,

\[ |bn - a| \leq |bn - bn| + |a_n - a|. \]

Now let \( N = \max(N_0, N_1) \). Then \( \forall n \geq N, n \geq N_0 \)

\[ |a_n - bn| < \frac{\varepsilon}{3} \quad \text{and} \quad |a_n - a| < \frac{\varepsilon}{3}. \]

Thus, \( |bn - a| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \).

So \( bn \) converges to \( a \) by definition of convergence.
To finish the proof (2 more lines from next page were): Consequently for any fixed $\varepsilon > 0$, $\forall n \in N$ where $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$
\[ |x - a_n| < \varepsilon \] and thus $\lim_{n \to \infty} a_n = x$.

(4c) Worth 2 pts.

(4e-1) I am doing (4b-1) or (4b-2), (circle one).

(4e-2) Check one box (1 pt for correct answer).

- I believe my proof is totally (or at least very close to being) correct.
- I believe my proof is a good start towards a correct proof but there are some mistakes/holes.
- I am just rambling in hopes of partial credit.
- Other. Explain:

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Space for (4b)

Let $\emptyset \neq A \subseteq \mathbb{R}$ and $A$ is bounded above. Define $\alpha = \text{lub}(A)$. Construct a sequence $\{a_n\}$ s.t. $\forall n \in N$, $a_n \in A$ and $\lim_{n \to \infty} a_n = \alpha$.

Case 1: Let $x \in A$. Then we define our sequence $\{a_n\}$ as $\forall n \in N$
\[ a_n = x. \] Since $x \in A$ clearly $\forall n$, $a_n \in A$. $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that $\forall n \geq N$
\[ |a_n - x| = |x - x| = 0 < \varepsilon \] and hence $\lim_{n \to \infty} a_n = x$.

Case 2: Let $x \notin A$. Then from (4b-2) we know $(\alpha - \varepsilon, \alpha) \cap A$ contains infinitely many points $y \geq 0$. We then define our sequence $\{a_n\}$ s.t. $\forall n \in N$, $a_n$ is some element of $(\alpha - \frac{1}{n}, \alpha) \cap A$. That is $a_n$ is a point selected from $(\alpha - \frac{1}{n}, \alpha) \cap A$, $a_n$ is some fixed element of $(\alpha - \frac{1}{n}, \alpha) \cap A$ and so on. And $\alpha \in (\alpha - \frac{1}{n}, \alpha) \cap A$ implies that $a_n \in A$.

For any fixed $\varepsilon > 0$, let $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$. Then $N \geq \frac{1}{\varepsilon}$ and hence $N \leq \varepsilon$.

Thus $\forall n \geq \frac{1}{\varepsilon} \Rightarrow (\alpha - \frac{1}{n}, \alpha) \subseteq (\alpha - \varepsilon, \alpha)$, and hence $\forall n \geq \frac{1}{\varepsilon}$ one $(\alpha - \varepsilon, \alpha)$. $a_n \in (\alpha - \varepsilon, \alpha)$ implies $\alpha - \varepsilon < a_n$ and $a_n < \alpha$. Thus $\alpha - a_n < \varepsilon$, since $a_n < \alpha$.

$a_n > 20$ and we can write $|a_n - x| < \varepsilon$ or $|a_n - x| < \varepsilon$. 

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#5 Countability

(5a) Fill-in blanks.

(5a-1) Two sets A and B are equivalent (i.e. have the same cardinality), denoted $A \sim B$, if and only if there exists a function $f : A \rightarrow B$ such that $f$ is bijective.

(5a-2) By definition, the set $A$ is countable if $A \sim \mathbb{N}$.

(5a-3) $\mathbb{Q}$ is countable.

(5a-4) $\mathbb{R}$ is uncountable.

(5a-4) If each $C_n$ is at most countable, then $\bigcup C_n$ is at most countable.

Fill-in the blanks of (5a-3) & (5a-4) with "uncountable, countable, atmost countable", using each choice once and only once.

(5b) Prove one, but not both, of (5b-1) and (5b-2). "Space." is provided next 2 pages.

Let $F = \{0, 1\} \subseteq \mathbb{R}$.

So $F$ is the set with two elements: the number 0 and the number 1.

One encounters the set $F$ often in probability classes: a trial (such as tossing a coin) with 2 possible outcomes (0 for heads, 1 for tails).

(5b-1) Let

$$C = \{ (e_1, e_2, \ldots, e_N) \in \mathbb{R}^N : N \in \mathbb{N} \text{ and each } e_j \in F \}.$$

Show $C$ is countable.

(5b-2) Let

$$D = \{ \{ e_j \}_{j=1}^{\infty} : \text{each } e_j \in F \}.$$

Show $D$ is uncountable.
(5c) Worth 2 points.

(5c-1) I am doing (5b-1) or (5b-2). (circle one).

(5c-2) Check one box.

☐ I believe my proof is totally (or at least very close to being) correct.
☐ I believe my proof is a good start towards a correct proof but there are some mistakes/holes.
☐ I am just rambling in hope of partial credit.
☐ Other, Explain:

______________________________

Hints on (5b)

Hints for (5b-1)

1. So C is the set of all (finite) n-tuples of 0's and 1's. Pictorially, you can think of C as:
   
   \[
   (0) \quad (1) \\
   (0,0), (0,1), (1,0), (1,1) \\
   (0,0,0), (0,0,1), (0,1,1), \ldots, (1,1,1)
   \]

2. You can use, without proving, (5a-4). Don't forget to show C is not finite.

Hints for (5b-2)

1. So D is the set of (infinite) sequences of 0's and 1's.
2. Outline of Pf.
   
   Step 1: D is not finite because \[\ldots\]. So D is infinite.
   Step 2: Assume D is cb. WTF \[\Rightarrow\]
   D cb so can enumerate \(D = \{x_n : n \in \mathbb{N}\}\) where \(\chi_n = \{e_1^n, e_2^n, e_3^n, \ldots, e_i^n, \ldots\}\).
   Now use a Cantor Diagonalization Argument to find \[\not\Leftarrow\].
Theorem: The set $C = \{e_1, e_2, \ldots, e_n\} \subset R^N$, with $e_i \in F$, is countable.

Proof: One could note that the function $f: C \rightarrow N$ defined by $f((e_1, e_2, \ldots, e_n)) = 2^n + \left(\sum e_i \cdot 2^{n-i}\right) - 1$ is a bijection.

Should we not wish to dive into the realm of binary representations, we could begin by noting that the subset $C_0 = \{e_0, (0,0), (0,0,0), (0,0,0,0), \ldots\}$ is infinite since $C_0 \cap N$ by the function which simply counts the numbers of zeroes in a given $n$-tuple. Since $C_0 \subseteq C$ and $C_0$ is infinite, $C$ must be infinite.

Next, note that $C = \bigcup_{e_i} \{e_1, e_2, \ldots, e_n\} : e_i \in F$. Since each $e_i$ can only have one of two values (there are exactly $2^n$ elements in $\{e_1, e_2, \ldots, e_n\} : e_i \in F$), as $2^n$ is just some finite number, $\bigcup_{e_i} \{e_1, e_2, \ldots, e_n\} : e_i \in F$ is the countable union of at most countable sets (since every finite set is at most countable) and by (5a-4) $C$ must therefore be at most countable. An at most countable set can either be finite or countable, and since we have shown $C$ is not finite, it therefore must be countable.
Proof. Let $F = \{0, 1\} \in \mathbb{R}$ and
\[ D = \{ \{e_i\}_{i=1}^{\infty} : \text{each } e_i \in F \} \]
We will show that $D$ is uncountable.

To see that $D$ is not finite, consider the function
\[ g : \mathbb{N} \rightarrow D \] where $g(n)$ is the sequence with 1 in the $n$th coordinate and 0 in the remaining coordinates, so
\[ g(n) = \{0, 0, \ldots, 0, 1, 0, 0, \ldots\} \]
Clearly $g$ is 1-to-1 so $D$ is not finite.

Now assume $D$ is countable. So we can enumerate $D$, say $D = \{x_n : n \in \mathbb{N}\}$ with
\[ x_n = \{\varepsilon_1^n, \varepsilon_2^n, \varepsilon_3^n, \ldots, \varepsilon_n^n, \ldots\} \]
Pictorially we have:
\[
\begin{align*}
x_1 &= \{\varepsilon_1^1, \varepsilon_2^1, \varepsilon_3^1, \varepsilon_4^1, \varepsilon_5^1, \ldots\} \\
x_2 &= \{\varepsilon_1^2, \varepsilon_2^2, \varepsilon_3^2, \varepsilon_4^2, \varepsilon_5^2, \ldots\} \\
x_3 &= \{\varepsilon_1^3, \varepsilon_2^3, \varepsilon_3^3, \varepsilon_4^3, \varepsilon_5^3, \ldots\} \\
x_4 &= \{\varepsilon_1^4, \varepsilon_2^4, \varepsilon_3^4, \varepsilon_4^4, \varepsilon_5^4, \ldots\} \\
x_5 &= \{\varepsilon_1^5, \varepsilon_2^5, \varepsilon_3^5, \varepsilon_4^5, \varepsilon_5^5, \ldots\} \\
\end{align*}
\]
Define $x = \{\varepsilon_i : i = 1, 2, 3, \ldots\}$ where
\[ \varepsilon_i = 1 - \varepsilon_i^1 \begin{cases} 1 & \text{if } \varepsilon_i^1 = 0 \\ 0 & \text{if } \varepsilon_i^1 = 1 \end{cases} \]
Clearly, $x \in D$. But for each $n \in \mathbb{N}$, the sequence $x$ and the sequence $x_n$ have different $n$th coordinates so $x_n \neq x$.

But $D = \{x_n : n \in \mathbb{N}\}$ and $x \neq x_n \forall n \in \mathbb{N}$ implies that $x \notin D$. This is a contradiction.

So $D$ is uncountable.