Problem 11. Variant of Exercise 1.1.10. §1.1 BS4p11

Let the function $f\colon \mathbb{R}\setminus\{1\}\to\mathbb{R}$ be defined by

$$f(x)=\frac{1}{x-1}.$$

Find the below preimages and (direct) images. Sets are in interval notation ⟨e.g., $(3,5]=\{x\in R\colon 3<x\leq 5\}$⟩.

Just type your solution directly into the statement of the problem. HINT. Graph the function.

1. $f\,[\,(3,5]\,] = ?$

2. $f\,[\,[-3,0)\,] = ?$

3. $f^{-1}\left[\,(\frac{1}{3},\frac{1}{2}]\,\right] = ?$

4. $f^{-1}\,[\,(-\infty,+\infty)\,] = ?$

5. $f^{-1}\,[\,[0,\infty)\,] = ?$

Problem 12. Variant of Exercise 1.1.15. Basically, prove (8) from the Functions and Sets handout. §1.1 BS4p11

Let $f\colon X\to Y$ be a function and $\{B_i\}_{i\in I}$ be a collection of sets of Y. Prove that

$$f^{-1}\left[\bigcup_{i\in I}B_i\right] \;=\; \bigcup_{i\in I}f^{-1}\left[B_i\right].$$

Use the definition of image and preimage. You may not use use (1)–(10) from Functions and Sets.

Proof. Your proof goes here. □

Problem 13. Variant of Exercise 1.1.14. Basically, prove (5) from the Functions and Sets handout. §1.1 BS4p11

Let $f\colon X\to Y$ be a function and $\{A_i\}_{i\in I}$ be a collection of sets of X. Prove that

$$f\left[\bigcup_{i\in I}A_i\right] \;=\; \bigcup_{i\in I}f\left[A_i\right].$$

Use the definition of image and preimage. You may not use use (1)–(10) from Functions and Sets.

Problem 14. Variant of Exercise 1.3.2c. Recall, the cardinality of a set T is the number of elements in T and is denoted/Latexed by $\mathrm{card}(T)$. ⟨In LaTeX, \textrm is short for "text roman"⟩. §1.3 BS4p22

Lemma 1. Let A_1 and A_2 are finite subsets of X. Then $A_1\cup A_2$ is finite and

$$\mathrm{card}(A_1\cup A_2) \;=\; \mathrm{card}(A_1)+\mathrm{card}(A_2\setminus A_1). \tag{$*$}$$

Lemma 1's Thinking Land. LTGBG. As is common, let's use $\biguplus$ to denote disjoint union. Then

$$A_1 \cup A_2 \;=\; A_1 \biguplus (A_2 \setminus A_1).$$

For any disjoint finite subsets A and B of a universal set U,

$$\operatorname{card}(A \cup B) \;=\; \operatorname{card}(A) + \operatorname{card}(B \setminus A).$$

So $(*)$ holds. □

Theorem 2. Let I be an infinite subset of the set X and F be a finite subset of X.

Prove that the set $I \setminus F$ is infinite by using **Lemma 1**.

WARNING. This exercise is to prove the book's Thm. 1.3.4c; thus, you may not use Thm. 1.3.4c.

Problem 15. Variant of Exercise 1.3.11. §1.3 BS4p22

For each of the below sets S_i, find:

a. the power set of S_i, with is denoted/LaTeXed $\mathcal{P}(S_i)$

b. the cardinality of the power set of S_i, which is denoted/LaTeXed $\operatorname{card}(\mathcal{P}(S_i))$.

(1) $S_1 = \{1\}$

(2) $S_2 = \{1, 2\}$

(3) $S_3 = \{1, 2, 3\}$

(4) $S_4 = \{1, 2, 3, 4\}$

Problem 16. Variant of Exercise 1.3.12. §1.3 BS4p22

Using induction to prove that if the set T has $n \in \mathbb{N}$ elements, then $\mathcal{P}(T)$ has 2^n elements.
