CHAPTER 1

Logic and Proofs

We recommend that you read the *Preface to the Student* before beginning this first chapter. Most of the terms and concepts in that *Preface* should be familiar to you, but it is well worth making sure you know the terminology and notations we will use throughout the book. It is especially important that you know precisely the definitions of such terms as: "divides," "prime," "rational," and "even" and "odd."

As described in the *Preface*, mathematics is concerned with the formation of a **theory** (collection of true statements) that describes patterns or relationships among quantities and structures. It is characterized by **deductive reasoning**, in which one uses logic to develop and extend a theory by drawing conclusions based on statements accepted as true. We give **proofs** to demonstrate that our conclusions are true. This chapter will provide a working knowledge of the basics of logic and how to construct a proof.

1.1 Propositions and Connectives

Our goal in this section is to understand truth values of propositions and how propositions can be combined using logical connectives.

Most sentences, such as "$\pi > 3$" and "Earth is the closest planet to the sun," have a truth value. That is, they are either true or false. We call these sentences propositions. Other sentences, such as "What time is it?" and "Look out!" are interrogatory or exclamatory; they express complete thoughts but have no truth value.

> **DEFINITION** A **proposition** is a sentence that has exactly one truth value: true, which we denote by T, or false, which we denote by F.

Some propositions, such as "$7^2 = 60$," have easily determined truth values. It will take years to determine the truth value of the proposition "The North Pacific right whale will be an extinct species before the year 2525." Other statements, such as "Euclid was left-handed," are propositions whose truth values may never be known.

Sentences like "She lives in New York City" and "$x^2 = 36$" are not propositions because each could be true or false depending upon the person to whom "she" refers and what numerical value is assigned to x. We will deal with sentences like these in Section 1.3.

The statement "This sentence is false" is not a proposition because it is neither true nor false. It is an example of a **paradox**—a situation in which, from premises that look reasonable, one uses apparently acceptable reasoning to derive a conclusion that seems to be contradictory. If the statement "This sentence is false" is true, then by its meaning it must be false. On the other hand, if the given statement is false, then what it claims is false, so it must be true. The study of paradoxes such as this has played a key role in the development of modern mathematical logic. A famous example of a paradox formulated in 1901 by Bertand Russell* is discussed in Section 2.1.

By applying logical connectives to propositions, we can form new propositions.

> **DEFINITION** The **negation** of a proposition P, denoted $\sim P$, is the proposition "not P." The proposition $\sim P$ is true exactly when P is false.

The truth value of the negation of a proposition is the opposite of the truth value of the proposition. For example, the negation of the false proposition "7 is divisible by 2" is the true statement "It is not the case that 7 is divisible by 2," or "7 is not divisible by 2."

> **DEFINITIONS** Given propositions P and Q, the **conjunction** of P and Q, denoted $P \wedge Q$, is the proposition "P and Q." $P \wedge Q$ is true exactly when *both* P and Q are true.
>
> The **disjunction** of P and Q, denoted $P \vee Q$, is the proposition "P or Q." $P \vee Q$ is true exactly when *at least one* of P or Q is true.

Examples. If C is the proposition "19 is composite" and M is "45 is a multiple of 3," we know C is false and M is true. Thus "19 is composite and 45 is a multiple of 3," written using logical connectives as $C \wedge M$, is a false proposition, while "19 is composite or 45 is a multiple of 3," which has form $C \vee M$, is true. The false proposition "Either 19 is composite or 45 is not a multiple of 3" has the form $C \vee \sim M$.

The English words *but*, *while*, and *although* are usually translated symbolically with the conjunction connective, because they have the same meaning as *and*. For

* Bertrand Russell (1872–1970) was a British philosopher, mathematician, and advocate for social reform. He was a strong voice for precision and clarity of arguments in mathematics and logic. He coauthored *Principia Mathematica* (1910–1913), a monumental effort to derive all of mathematics from a specific set of axioms and well-defined rules of inference.

example, we would write "19 is not composite, but 45 is a multiple of 3" in symbolic form as: $(\sim C) \wedge M$.

An important distinction must be made between a statement and the *form* of a statement. In the previous example "19 is composite and 45 is a multiple of 3" is a proposition with truth value F. We used the form $C \wedge M$ to represent this proposition, but *the form $C \wedge M$ itself has no truth value* unless C and M are assigned to be specific propositions. If we let C be "Copenhagen is the capital of Denmark" and M be "Madrid is the capital of Spain," then $C \wedge M$ would have the value T.

To repeat: a propositional form does not have *a* truth value. Instead, each form has a *list* of truth values that depend on the values assigned to its components. This list is displayed by presenting all possible combinations for the truth values of its components in a truth table. Since the connectives $\wedge$ and $\vee$ involve two components, their truth tables must list the four possible combinations of the truth values of those components:

P	Q	$P \wedge Q$
T	T	T
F	T	F
T	F	F
F	F	F

P	Q	$P \vee Q$
T	T	T
F	T	T
T	F	T
F	F	F

Since the value of $\sim P$ depends only on the two possible values for P, its truth table is

P	$\sim P$
T	F
F	T

Frequently you will encounter compound propositions formed from more than two propositional variables. The propositional form $(P \wedge Q) \vee \sim R$ has three variables P, Q, and R; it follows that there are $2^3 = 8$ possible combinations of truth values. The two main components are $P \wedge Q$ and $\sim R$. We make truth tables for these and combine them by using the truth table for $\vee$.

P	Q	R	$P \wedge Q$	$\sim R$	$(P \wedge Q) \vee \sim R$
T	T	T	T	F	T
F	T	T	F	F	F
T	F	T	F	F	F
F	F	T	F	F	F
T	T	F	T	T	T
F	T	F	F	T	T
T	F	F	F	T	T
F	F	F	F	T	T

The statement "Either 7 is prime and 9 is even or else 11 is not less than 3" may be symbolized by $(P \wedge Q) \vee \sim R$, where P is "7 is prime," Q is "9 is even," and R

is "11 is less than 3." We know P is true, Q is false and R is false. Therefore, $(P \wedge Q)$ is false and $\sim R$ is true. Thus $(P \wedge Q) \vee \sim R$ is true, in agreement with line 7 of the table. Thus the proposition "Either 7 is prime and 9 is even or else 11 is not less than 3" is a true statement.

Some compound forms always yield the value true just because of the way they are formed; others are always false.

> **DEFINITIONS** A **tautology** is a propositional form that is true for every assignment of truth values to its components.
>
> A **contradiction** is a propositional form that is false for every assignment of truth values to its components.

For example, the *Law of Excluded Middle*, $P \vee \sim P$, is a tautology because $P \vee \sim P$ is true when P is true and true when P is false. We know that a statement like "The absolute value function is continuous or it is not continuous" must be true because it has the form of the Law of Excluded Middle.

Example. Show that $(P \vee Q) \vee (\sim P \wedge \sim Q)$ is a tautology.

The truth table for this propositional form is

P	Q	$P \vee Q$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$	$(P \vee Q) \vee (\sim P \wedge \sim Q)$
T	T	T	F	F	F	T
F	T	T	T	F	F	T
T	F	T	F	T	F	T
F	F	F	T	T	T	T

Since the last column is all true, $(P \vee Q) \vee (\sim P \wedge \sim Q)$ is a tautology.

Both $\sim(P \vee \sim P)$ and $Q \wedge \sim Q$ are examples of contradictions. The negation of a contradiction is, of course, a tautology.

Writing a proof requires the ability to connect statements so that the truth of any given statement in the proof follows logically from previous statements in the proof, from known results, or from basic assumptions. Particularly important is the ability to recognize or write a statement equivalent to another. Sometimes, it is the *form* of a compound statement that may be used to find a useful equivalent.

> **DEFINITION** Two propositional forms are **equivalent** if and only if they have the same truth tables.

Example. The propositional forms P and $\sim(\sim P)$ are equivalent. The truth tables for these forms may be combined in one table to show that they are the same:

P	$\sim P$	$\sim(\sim P)$
T	F	T
F	T	F

The fact that P and $\sim(\sim P)$ have the same truth value for each line of the truth table means that whatever proposition we choose for P, the truth value of P and $\sim(\sim P)$ are identical.

Some of the most commonly used equivalent forms are presented in the following theorem.

Theorem 1.1.1 For propositions P, Q, and R, the following are equivalent:

(a)	P	and	$\sim(\sim P)$	Double Negation Law
(b)	$P \vee Q$	and	$Q \vee P$	Commutative Laws
(c)	$P \wedge Q$	and	$Q \wedge P$	
(d)	$P \vee (Q \vee R)$	and	$(P \vee Q) \vee R$	Associative Laws
(e)	$P \wedge (Q \wedge R)$	and	$(P \wedge Q) \wedge R$	
(f)	$P \wedge (Q \vee R)$	and	$(P \wedge Q) \vee (P \wedge R)$	Distributive Laws
(g)	$P \vee (Q \wedge R)$	and	$(P \vee Q) \wedge (P \vee R)$	
(h)	$\sim(P \wedge Q)$	and	$\sim P \vee \sim Q$	DeMorgan's* Laws
(i)	$\sim(P \vee Q)$	and	$\sim P \wedge \sim Q$	

Proof.

(a) See the discussion above.

(h) By examining the fourth and seventh columns of their combined truth tables as shown here,

P	Q	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim P$	$\sim Q$	$\sim P \vee \sim Q$
T	T	T	F	F	F	F
F	T	F	T	T	F	T
T	F	F	T	F	T	T
F	F	F	T	T	T	T

we see that the truth tables for $\sim(P \wedge Q)$ and $\sim P \vee \sim Q$ are identical. Thus $\sim(P \wedge Q)$ and $\sim P \vee \sim Q$ are equivalent propositional forms.

Proofs of the remaining parts are left as exercises. ■

In addition to making tables to verify the remaining parts of Theorem 1.1.1, you should also think about why two propositional forms are equivalent by looking

* Augustus DeMorgan (1806–1871) was an English logician and mathematician whose contributions include his notational system for symbolic logic. He also introduced the term "mathematical induction" (see Section 2.4) and developed a rigorous foundation for that proof technique.

at their meanings. For part (h), negation is applied to a conjunction. The form $\sim(P \wedge Q)$ is true precisely when $P \wedge Q$ is false. This happens when one of P or Q is false, or in other words, when one of $\sim P$ or $\sim Q$ is true. Thus, $\sim(P \wedge Q)$ is equivalent to $\sim P \vee \sim Q$. That is, to say "You don't have both P and Q" is the same as saying "You don't have P or you don't have Q."

As an example of how this theorem might be useful in dealing with statements, suppose we are told that the statement "The function f is increasing and concave upward" is false. The statement has the form $P \wedge Q$, where P is the statement "f is increasing" and Q is the statement "f is concave upward." The negation $\sim(P \wedge Q)$ is "It is not the case that f is increasing and f is concave upward." By part (h) above, this is equivalent to $\sim P \vee \sim Q$, which is

> "It is not the case that f is increasing or it is not the case that f is concave upward."

An easier way to say this is

"f is not increasing or f is not concave upward."

A **denial** of a proposition P is any proposition equivalent to $\sim P$. A proposition has only one negation, $\sim P$, but always has many denials, including $\sim P$, $\sim\sim\sim P$, $\sim\sim\sim\sim\sim P$, etc. DeMorgan's Laws provide others ways to construct useful denials.

Example. A denial of "Either Miss Scarlet is not guilty or the crime did not take place in the ballroom" is "The crime took place in the ballroom and Miss Scarlet is guilty." This can be verified by writing the two propositions symbolically as $(\sim S) \vee (\sim B)$ and $B \wedge S$, respectively, and checking that their truth tables have exactly opposite values. We could also observe that $B \wedge S$ is equivalent to $S \wedge B$ so a denial of $B \wedge S$ is equivalent to $\sim(S \wedge B)$, which we know by DeMorgan's Laws is equivalent to $(\sim S) \vee (\sim B)$.

Example. The statement "Line L_1 has slope 3/5 or line L_2 does not have slope -4" may be symbolized using the form $P \vee \sim Q$, so its negation is $\sim(P \vee \sim Q)$. We can write a simpler denial $(\sim P) \wedge Q$ by applying DeMorgan's Laws and the Double Negation Law. The simplified denial says "Line L_1 does not have slope $3/5$ and line L_2 has slope -4."

Notice that someone might read the negation $\sim(P \vee \sim Q)$ as "It is not the case that L_1 has slope $3/5$ or line L_2 does not have slope -4." This sentence is ambiguous because without some further explanation, it is not clear if the phrase "It is not the case" refers to the entire remainder of the sentence or to just "L_1 has slope $3/5$."

Ambiguities like the one above are sometimes allowable in English but can cause trouble in mathematics. To avoid ambiguities, you should use delimiters, such as parentheses ( ), square brackets [ ], and braces { }.

To avoid writing large numbers of delimiters, we use the following rules, which we refer to as the *hierarchy of connectives*.

> First, $\sim$ always is applied to the smallest proposition following it.
> Then, $\wedge$ always connects the smallest propositions surrounding it.
> Finally, $\vee$ connects the smallest propositions surrounding it.

Thus, $\sim P \vee Q$ is an abbreviation for $(\sim P) \vee Q$, but $\sim(P \vee Q)$ is the only way to write the negation of $P \vee Q$. Here are some other examples:

$P \vee Q \wedge R$ abbreviates $P \vee (Q \wedge R)$.
$P \wedge \sim Q \vee \sim R$ abbreviates $[P \wedge (\sim Q)] \vee (\sim R)$.
$\sim P \vee \sim Q$ abbreviates $(\sim P) \vee (\sim Q)$.
$\sim P \wedge \sim R \vee \sim P \wedge R$ abbreviates $[(\sim P) \wedge (\sim R)] \vee [(\sim P) \wedge R]$.

When the same connective is used several times in succession, parentheses may be omitted. We reinsert parentheses from the left, so that $P \vee Q \vee R$ is really $(P \vee Q) \vee R$. For example, $R \wedge P \wedge \sim P \wedge Q$ abbreviates $[(R \wedge P) \wedge (\sim P)] \wedge Q$, whereas $R \vee P \wedge \sim P \vee Q$, which does not use the same connective consecutively, abbreviates $(R \vee [P \wedge (\sim P)]) \vee Q$. Leaving out parentheses is not required; some propositional forms are much easier to read with a few well-chosen "unnecessary" parentheses.

Exercises 1.1

1. Use your knowledge of number systems to determine whether each is true or false.
   (a) 11 is a rational number.
   ★ (b) 5π is a rational number.
   (c) There are exactly 3 prime numbers between 40 and 50.
   (d) There are exactly 5 prime numbers less than 10.
   (e) 29 is a composite number.
   (f) 0 is a natural number.
   ★ (g) $(5 + 2i)(5 - 2i)$ is a real number.
   (h) 18 is a multiple of 12.
2. Which of the following are propositions? Give the truth value of each proposition.
   (a) What time is dinner?
   (b) It is not the case that $5 + \pi$ is not a rational number.
   ★ (c) $x/2$ is a rational number.
   (d) $2x + 3y$ is a real number.
   (e) Either $3 + \pi$ is rational or $3 - \pi$ is rational.
   ★ (f) Either 2 is rational and π is irrational, or 2π is rational.
   (g) Either 5π is rational and 4.9 is rational, or 3π is rational.
   (h) $-\frac{1}{2}$ is rational, and either $3\pi < 10$ or $3\pi > 15$.
   (i) It is not the case that 39 is prime, or that 64 is a power of 2.
   (j) There are more than three false statements in this book and this statement is one of them.
3. Make truth tables for each of the following propositional forms.
   ★ (a) $P \wedge \sim P$.
   (b) $P \vee \sim P$.
   ★ (c) $P \wedge \sim Q$.
   (d) $P \wedge (Q \vee \sim Q)$.
   ★ (e) $(P \wedge Q) \vee \sim Q$.
   (f) $\sim(P \wedge Q)$.
   (g) $(P \vee \sim Q) \wedge R$.
   (h) $\sim P \wedge \sim Q$.

★ (i) $P \wedge (Q \vee R)$. (j) $(P \wedge Q) \vee (P \wedge R)$.
(k) $P \wedge P$. (l) $(P \wedge Q) \vee (R \wedge \sim S)$.

4. If P, Q, and R are true while S and T are false, which of the following are true?
★ (a) $Q \wedge (R \wedge S)$. (b) $Q \vee (R \wedge S)$.
★ (c) $(P \vee Q) \wedge (R \vee S)$. (d) $(\sim P \vee \sim Q) \vee (\sim R \vee \sim S)$.
(e) $\sim P \vee \sim Q$. ★ (f) $(\sim Q \vee S) \wedge (Q \vee S)$.
★ (g) $(P \vee S) \wedge (P \vee T)$.

5. Use truth tables to prove the remaining parts of Theorem 1.1.1.

6. Which of the following pairs of propositional forms are equivalent?
★ (a) $P \wedge P, P$. (b) $P \vee P, P$.
★ (c) $P \wedge Q, Q \wedge P$. (d) $(\sim P) \vee (\sim Q), \sim(P \vee \sim Q)$.
★ (e) $\sim P \wedge \sim Q, \sim(P \wedge \sim Q)$. (f) $\sim(P \wedge Q), \sim P \wedge \sim Q$.
★ (g) $(P \wedge Q) \vee R, P \wedge (Q \vee R)$. (h) $(P \wedge Q) \vee R, P \vee (Q \wedge R)$.

7. Determine the propositional form and truth value for each of the following:
(a) It is not the case that 2 is odd.
(b) $f(x) = e^x$ is increasing and concave up.
(c) Both 7 and 5 are factors of 70.
(d) Perth or Panama City or Pisa is located in Europe.

8. P, Q, and R are propositional forms, and P is equivalent to Q, and Q is equivalent to R. Prove that
★ (a) Q is equivalent to P.
(b) P is equivalent to R.
(c) $\sim Q$ is equivalent to $\sim P$.

9. Use a truth table to determine whether each of the following is a tautology, a contradiction, or neither.
(a) $(P \wedge Q) \vee (\sim P \wedge \sim Q)$.
(b) $\sim(P \wedge \sim P)$.
★ (c) $(P \wedge Q) \vee (\sim P \vee \sim Q)$.
(d) $(A \wedge B) \vee (A \wedge \sim B) \vee (\sim A \wedge B) \vee (\sim A \wedge \sim B)$.
(e) $(Q \wedge \sim P) \wedge \sim(P \wedge R)$.
(f) $P \vee [(\sim Q \wedge P) \wedge (R \vee Q)]$.

10. Suppose A is a tautology and B is a contradiction. Are the following tautologies, contradictions, or neither?
★ (a) $A \wedge B$. (b) $A \wedge \sim B$.
★ (c) $A \vee B$. (d) $\sim(\sim A \wedge B)$.

11. Give a useful denial of each statement.
★ (a) x is a positive integer. (Assume that x is some fixed integer.)
(b) Cleveland will win the first game or the second game.
★ (c) $5 \geq 3$.
(d) 641,371 is a composite integer.
★ (e) Roses are red and violets are blue.
(f) T is not bounded or T is compact. (Assume that T is a fixed object.)
(g) M is odd and one-to-one. (Assume that M is some fixed function.)

(h) The function f has positive first and second derivatives at x_0. (Assume that f is a fixed function and x_0 is a fixed real number.)

(i) The function g has a relative maximum at $x = 2$ or $x = 4$ and a relative minimum at $x = 3$. (Assume that g is a fixed function.)

(j) Neither $z < s$ nor $z \leq t$ is true. (Assume that z, s, and t are fixed real numbers.)

(k) R is transitive but not reflexive. (Assume that R is a fixed object.)

12. Restore parentheses to these abbreviated propositional forms.

(a) $\sim\sim P \vee \sim Q \wedge \sim S$.

(b) $Q \wedge \sim S \vee \sim(\sim P \wedge Q)$.

(c) $P \wedge \sim Q \vee \sim P \wedge \sim R \vee \sim P \wedge S$.

(d) $\sim P \vee Q \wedge \sim\sim P \wedge Q \vee R$.

13. Other logical connectives between two propositions P and Q are possible.

(a) The word *or* is used in two different ways in English. We have presented the truth table for $\vee$, the **inclusive or,** whose meaning is "one or the other or both." The **exclusive or,** meaning "one or the other but not both" and denoted $\veebar$, has its uses in English, as in "She will marry Heckle or she will marry Jeckle." The "inclusive or" is much more useful in mathematics and is the accepted meaning unless there is a statement to the contrary.

★ (i) Make a truth table for the "exclusive or" connective $\veebar$.

(ii) Show that $A \veebar B$ is equivalent to $(A \vee B) \wedge \sim(A \wedge B)$.

(b) "NAND" and "NOR" circuits are commonly used as a basis for flash memory chips. A NAND B is defined to be the negation of "A and B." A NOR B is defined to be the negation of "A or B."

(i) Write truth tables for NAND and NOR connectives.

(ii) Show that (A NAND B) $\vee$ (A NOR B) is equivalent to (A NAND B).

(iii) Show that (A NAND B) $\wedge$ (A NOR B) is equivalent to (A NOR B).

1.2 Conditionals and Biconditionals

Sentences of the form "If P, then Q" are the most important kind of propositions in mathematics. You have seen many examples of such statements in mathematics courses: from precalculus, "If two lines in a plane have the same slope, then the lines are parallel"; from trigonometry, "If $\sec\theta = \frac{5}{3}$, then $\sin\theta = \frac{4}{5}$."; from calculus, "If f is differentiable at x_0 and $f(x_0)$ is a relative minimum for f, then $f'(x_0) = 0$."

DEFINITIONS For propositions P and Q, the **conditional sentence** $P \Rightarrow Q$ is the proposition "If P, then Q." Proposition P is called the **antecedent** and Q is the **consequent.** The conditional sentence $P \Rightarrow Q$ is true if and only if P is false or Q is true.