Welcome to the study of mathematical reasoning. The authors know that many students approach this material with some apprehension and uncertainty. Some students feel that "This isn't like other mathematics courses," or expect that the study of proofs is something they won't really have to do or won't use later. These feelings are natural as you move from calculation-oriented courses where the goals emphasize performing computations or solving certain equations, to more advanced courses where the goal may be to establish whether a mathematical structure has certain properties. This textbook is written to help ease the transition between these courses. Let's consider several questions students commonly have at the beginning of a "transition" course.

## Why write proofs?

Mathematicians often collect information and make observations about particular cases or phenomena in an attempt to form a theory (a model) that describes patterns or relationships among quantities and structures. This approach to the development of a theory uses **inductive reasoning**. However, the characteristic thinking of the mathematician is **deductive reasoning**, in which one uses logic to develop and extend a theory by drawing conclusions based on statements accepted as true. Proofs are essential in mathematical reasoning because they demonstrate that the conclusions are true. Generally speaking, a mathematical explanation for a conclusion has no value if the explanation cannot be backed up by an acceptable proof.

## Why not just test and repeat enough examples to confirm a theory?

After all, as is typically done in natural and social sciences, the test for truth of a theory is that the results of an experiment conform to predictions, and that when the experiment is repeated under the same circumstances the result is always the same. The difference is that in mathematics we need to know whether a given statement is *always* true, so while the statement may be true for many (even infinitely many) examples, we would never know whether another example might show the statement to be false. By studying examples, we might conclude that the statement

$$\text{“}x^2 - 3x + 43 \text{ is a prime number”}$$

is true for all positive integers $x$. We could reach this conclusion testing the first 10 or 20 or even the first 42 integers $1, 2, 3, \ldots, 42$. In each of these cases and others, such as 44, 45, 47, 48, 49, 50 and more, $x^2 - 3x + 43$ *is* a prime number. But the statement is not always true because $43^2 - 3(43) + 43 = 1763$, which is $41 \cdot 43$. Checking examples is helpful in gaining insight for understanding concepts and relationships in mathematics, but is not a valid proof technique unless we can somehow check all examples.

## Why not just rely on proofs that someone else has done?

One answer follows from the statement above that deductive reasoning characterizes the way mathematicians think. In the sciences, a new observation may force a complete rethinking of what was thought to be true; in mathematics what we know to be true (by proof) is true forever unless there was a flaw in the reasoning. By learning the techniques of reasoning and proof, you are learning the tools of the trade.

The first goal of this text is to examine standard proof techniques, especially concentrating on how to get started on a proof, and how to construct correct proofs using those techniques. You will discover how the logical form of a statement can serve as a guide to the structure of a proof of the statement. As you study more advanced courses, it will become apparent that the material in this book is indeed fundamental and the knowledge gained will help you succeed in those courses. Moreover, many of the techniques of reasoning and proof that may seem so difficult at first will become completely natural with practice. In fact, the reasoning that you will study is the essence of advanced mathematics and the ability to reason abstractly is a primary reason why applicants trained in mathematics are valuable to employers.

## What am I supposed to know before beginning Chapter 1?

The usual prerequisite for a transition course is at least one semester of calculus. We will sometimes refer to topics that come from calculus and earlier courses (for example, differentiable functions or the graph of a parabola), but we won't be solving equations or finding derivatives.

You will need a good understanding of the basic concepts and notations from earlier courses. The list of definitions and relationships below includes the main things **you will need to have ready for immediate use** at any point in the text.

Be aware that definitions in mathematics, however, are not like definitions in ordinary English, which are based on how words are typically used. For example, the ordinary English word "cool" came to mean something good or popular when many people used it that way, not because it has to have that meaning. If people stop using the word that way, this meaning of the word will change. Definitions in mathematics have precise, fixed meanings. When we say that an integer is odd, we do not mean that it's strange or unusual. Our definition below tells you exactly what odd means. You may form a concept or a mental image that you may use to help understand (such as "ends in 1, 3, 5, 7, or 9"), but the mental image you form is not what has been defined. For this reason, definitions are usually stated with the "if and only if" connective because they describe exactly—no more, no less—the condition(s) to meet the definition.

## Sets

A **set** is a collection of objects, called the **elements**, or members of the set. When the object $x$ is in the set $A$, we write $x \in A$; otherwise $x \notin A$. The set $K = \{6, 7, 8, 9\}$ has four elements; we see that $7 \in K$ but $3 \notin K$. We may use set-builder notation to write the set $K$ as

$$\{x: x \text{ is an integer greater than 5 and less than 10}\},$$

which we read as "the set of $x$ such that $x$ is . . ." Observe that the set whose only element is 5 is not the same as the number 5; that is, $\{5\} \neq 5$. The **empty set** $\varnothing$ is a set with no elements.

We say that $A$ is a **subset** of $B$, and write $A \subseteq B$, if and only if every element of $A$ is an element of $B$. If sets $A$ and $B$ have exactly the same elements, we say they are **equal** and write $A = B$.

We use these notations for the number systems:

$\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers.
$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers.
$\mathbb{Q}$ is the set of all rational numbers.
$\mathbb{R}$ is the set of all real numbers.
$\mathbb{C}$ is the set of all complex numbers.

A set is **finite** if it is empty or if it has $n$ elements for some natural number $n$. Otherwise it is **infinite**. Thus the set $\{6, 7, 8, 9\}$ is finite. All the number systems listed above are infinite.

## The Natural Numbers

The properties below describe the basic arithmetical and ordering structure of the set $\mathbb{N}$.

1. *Successor properties*
   1 is a natural number.
   Every natural number $x$ has a unique successor $x + 1$.
   1 is not the successor of any natural number.

2. *Closure properties*
   The sum of two natural numbers is a natural number.
   The product of two natural numbers is a natural number.
3. *Associativity properties*
   For all $x, y, z \in \mathbb{N}$, $x + (y + z) = (x + y) + z$.
   For all $x, y, z \in \mathbb{N}$, $x(yz) = (xy)z$.
4. *Commutativity properties*
   For all $x, y \in \mathbb{N}$, $x + y = y + x$.
   For all $x, y \in \mathbb{N}$, $xy = yx$.
5. *Distributivity properties*
   For all $x, y, z \in \mathbb{N}$, $x(y + z) = xy + xz$.
   For all $x, y, z \in \mathbb{N}$, $(y + z)x = yx + zx$.
6. *Cancellation properties*
   For all $x, y, z \in \mathbb{N}$, if $x + z = y + z$, then $x = y$.
   For all $x, y, z \in \mathbb{N}$, if $xz = yz$, then $x = y$.

For natural numbers $a$ and $b$ we say $a$ **divides** $b$ (or $a$ is a **divisor** of $b$, or $b$ is a **multiple** of $a$) if and only if there is a natural number $k$ such that $b = ak$. For example, 7 divides 56 because there is a natural number (namely 8) such that $56 = 7 \cdot 8$.

A natural number $p$ is **prime** if and only if $p$ is greater than 1 and the only natural numbers that divide $p$ are 1 and $p$. A **composite** is a natural number that is neither 1 nor prime.

## The Fundamental Theorem of Arithmetic:

Every natural number larger than 1 is prime or can be expressed uniquely as a product of primes. For example, 440 can be expressed as $440 = 2^3 \cdot 5 \cdot 11$. If we list the prime factors in increasing order, then there is only one prime factorization: the primes and their exponents are uniquely determined.

## The Integers

The integers share properties 2 through 6 listed above for $\mathbb{N}$ (with the exception that we can't cancel $z = 0$ from the product $xz = yz$). Other important properties are:

For all $x$ in $\mathbb{Z}$, $x + 0 = 0$, $x \cdot 0 = 0$ and $x + (-x) = 0$.
For all $x, y, z$ in $\mathbb{Z}$, if $x < y$ and $z > 0$, $xy < yz$.
The product of two positive or two negative integers is positive; the product of a positive and a negative is negative.

The natural numbers and integers provide excellent settings for developing an understanding of the structure of a correct proof, so we will use the following definitions extensively in early examples of proof writing. In those proofs we make use of the properties of number systems and the fact that every integer is either even or odd, but not both.

An integer $x$ is **even** if and only if there is an integer $k$ such that $x = 2k$. An integer $x$ is **odd** if and only if there is an integer $j$ such that $x = 2j + 1$. For integers $a$ and $b$ with $a \neq 0$ we say $a$ **divides** $b$ if and only if there is an integer $k$ such that $b = ak$.

## Real and Rational Numbers

We think of the real numbers as being all the numbers along the number line. Each real number can be represented as an integer together with a finite or infinite decimal part. We use the standard notations for intervals on the number line. For real numbers $a$ and $b$ with $a < b$:

$(a, b) = \{x: x \in \mathbb{R} \text{ and } a < x < b\}$ is the **open interval from *a* to *b***.
$[a, b] = \{x: x \in \mathbb{R} \text{ and } a \leq x \leq b\}$ is the **closed interval from *a* to *b***.
$(a, \infty) = \{x: x \in \mathbb{R} \text{ and } a < x\}$ and $(-\infty, b) = \{x: x \in \mathbb{R} \text{ and } x < b\}$ are **open rays.**
$[a, \infty) = \{x: x \in \mathbb{R} \text{ and } a \leq x\}$ and $(-\infty, b] = \{x: x \in \mathbb{R} \text{ and } x \leq b\}$ are **closed rays.**

Note that the infinity symbol "$\infty$" is simply a notational convenience and does not represent any real number. Also, one should be careful not to confuse $(1, 6)$ with $\{2, 3, 4, 5\}$, since $(1, 6)$ is the set of all real numbers between 1 and 6 and contains, for example, $2, \pi, \sqrt{13}$, and $\frac{27}{5}$.

The real number $x$ is **rational** if and only if there are integers $p$ and $q$, with $q \neq 0$, such that $x = p/q$.

The rationals are exactly the numbers along the number line that have terminating or repeating decimal expressions. All other real numbers are **irrational**. In Chapter 1 we will see a proof that $\sqrt{2}$ is irrational. The number systems $\mathbb{R}$ and $\mathbb{Q}$ share many of the arithmetic and ordering properties of the naturals and integers, along with a new property:

Every number $x$ except 0 has a multiplicative inverse; that is, there is a number $y$ such that $xy = 1$.

## Complex Numbers

A complex number has the form $a + bi$, where $a$ and $b$ are real numbers and $i = \sqrt{-1}$. The **conjugate** of $a + bi$ is $a - bi$ and $(a + bi)(a - bi) = a^2 + b^2$. The set of reals is a subset of the complex numbers because any real number $x$ may be written as $x + 0i$. Complex numbers do not share the ordering properties of the reals.

## Functions

A **function** (or a **mapping**) is a rule of correspondence that associates to each element in a set $A$ a unique element in a second set $B$. No restriction is placed on the sets $A$ and $B$, which may be sets of numbers, or functions, or vegetables. To denote that $f$ is a function from $A$ to $B$, we write

$$f: A \to B$$

and say "$f$ maps $A$ to $B$." If $a \in A$ and the corresponding element of $B$ is $b$, we write

$$f(a) = b.$$

The elements of $A$ are sometimes called the **arguments** or **inputs** of the function. If $f(a) = b$ we say that $b$ is the **image** of $a$, or $b$ is the **value** of the function $f$ at $a$. We also say that $a$ is a **pre-image** of $b$.

For example, $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 1$ represents the correspondence that assigns to each real number $x$ the number that is one more than the square of $x$. The image of the real number 2 is 5 and $-3$ is a pre-image of 10.

The features that make $f$ a function from $A$ to $B$ are that every element of $A$ must have an image, that image must be in $B$, and most importantly, that no element of $A$ has more than one image. It is this **single-valued** property that make functions so useful.

If $f\colon A \to B$, the set $A$ is the **domain** of $f$, denoted $\mathrm{Dom}(f)$, and $B$ is the **codomain** of $f$. The set

$$\mathrm{Rng}(f) = \{f(x)\colon x \in A\}$$

of all images under the function $f$ is called the **range** of $f$. The range of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 1$ is $[1, \infty)$.

It is sometimes convenient to describe a function by giving only a domain and a rule. For functions whose domains and codomains are subsets of $\mathbb{R}$, the domain is sometimes left unspecified and assumed to be the largest possible subset of $\mathbb{R}$ for which image values may be obtained. With this assumption, the domain of $g(x) = \sqrt{x+1}$ is $[-1, \infty)$, because this is the largest set of real numbers for which $\sqrt{x+1}$ may be calculated.

When we say that $f\colon A \to B$, it is required that $\mathrm{Rng}(f) \subseteq B$. However, it may be that some elements of the codomain are not images under the function $f$; that is, the set $\mathrm{Rng}(f)$ may not be equal to $B$. In the special case when the range of $f$ *is* equal to $B$, we say $f$ maps $A$ **onto** $B$. It may also be that two different elements of $A$ have the same image in $B$. In the special case when any two different arguments have different images, we say that $f$ is **one-to-one**. Because the range of $f(x) = x^2 + 1$ is $[1, \infty)$, $f$ is not onto $\mathbb{R}$. Since $f(3)$ and $f(-3)$ have value 10, $f$ is not one-to-one.

## What am I allowed to assume for a proof?

You may be given specific instructions for some proof writing exercises, but generally the idea is that you may use what someone studying the topic of your proof would know. That is, when we prove something about intersecting lines we might use facts about the slope of a line, but we probably would not use properties of derivatives. This really is not much of a problem, except for our first proof examples, which deal with elementary concepts such as even and odd (because they provide meaningful examples and a familiar context in which to study logic and reasoning). For these proofs we are allowed to use the properties of integers and natural numbers that we already know *except* what we already know about evenness and oddness.

Remember, we don't expect you to become an expert at proving theorems overnight. With practice—studying lots of examples and exercises—the skills will come. Our goal is to help you write and think as mathematicians do, and to present a solid foundation in material that is useful in advanced courses. We hope you enjoy it.

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