Practice Problems from Section 3.6. Exercises 3.6.1–3.6.13 except for: 2, 10, 13.
These *Practice Problems* are a sampling of the type of problems which could be on the exam.
These *Practice Problem* are, in no way, meant as a comprehensive review for the exam.

Math is not a spectator sport.
Often we learn more from our failed attempts at a proof rather than immediately looking at the hints or reading a clean proof.
So give these problems a solid attempt before seeking help, e.g.:
looking through your notes and/or book, looking at the below hints, or looking at Piazza.
Since these problems are not to hand in, on Piazza you may share:
hints and/or an attempt at a solution for others to comment on.

Some hints are very generous. Do not except such generous hints on the exam.

1. Prove that each point on or inside the circle whose equation is

$$(x-1)^2 + (y-2)^2 = 4$$

is also inside the circle whose equation is

$$x^2 + y^2 = 26 \ .$$

hint. **Definition**. The point $(x_0, y_0) \in \mathbb{R}^2$ is:

- inside the circle whose equation is $(x-h)^2 + (y-k)^2 = r^2$ provided $(x_0-h)^2 + (y_0-k)^2 < r^2$
- on the circle whose equation is $(x-h)^2 + (y-k)^2 = r^2$ provided $(x_0-h)^2 + (y_0-k)^2 = r^2$
- outside the circle $(x-h)^2 + (y-k)^2 = r^2$ provided $(x_0-h)^2 + (y_0-k)^2 > r^2$.

hint. Symbolically looks: $\left(\forall\, (x,y) \in \mathbb{R}^2 \right) \; [\, P\,(x,y) \implies Q\,(x,y)\,]$ where $P\,(x,y)$ and $Q\,(x,y)$ are open senences in the variable x and y.

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3. Are the following statements true or false? Justify your answer.

3.1. For each integer a, if 3 does not divide a, then 3 divides $2a^2+1$.

3.2. For each integer a, if 3 divides $2a^2+1$, then 3 does not divide a.

3.3. For each integer a, we have 3 does not divide a if and only if 3 divides $2a^2+1$.

hint. Each one is true.

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4. Prove that if $x \in \mathbb{R}$ and y is a irrational number then $x+y$ is irrational or $x-y$ is irrational.

hint. Since being irrational does not give us much of *a bird in the hand*, can we somehow work with the rational numbers, which give us *some birds in the hand*? When symbolically write, note we can get a conditional open sentence by using $\mathbb{R}^2$ as the universe. Think about all them nice closure properties of $\mathbb{Q}$ (to help avoid algebra). Proof is very similar to proof of book's Prop. 3.19 (§3.3, p123).

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5. Prove that there exist irrational numbers x and y such that x^y is a rational number.

hint. We know $\sqrt{2}$ is irrational (by Thm. 3.20, p124). Try cases. Consider the number $\left(\sqrt{2}\right)^{\sqrt{2}}$, which is either rational or irrational (who knows, but who cares since ...) Case 1: The $\left(\sqrt{2}\right)^{\sqrt{2}}$ is rational. Case 2: The $\left(\sqrt{2}\right)^{\sqrt{2}}$ is irrational.

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6. Let a and b be natural numbers such that

$$a^2 \;=\; b^3 \ . \tag{6}$$

6.1. Prove that if a is even, then 4 divides a.

6.2. Prove that if 4 divides a, then 4 divides b.

6.3. Prove that if 4 divides b, then 8 divides a.

6.4. Prove that if a is even, then 8 divides a.

6.5. Give an example of natural numbers a and b such that a is even and $a^2 = b^3$, but b is not divisible by 8.

hint. Helpful. If $n \in \mathbb{N}$ then $\left[\, n \text{ is even} \overset{\text{Thm 3.10}}{\Longleftrightarrow} n^2 \text{ is even} \,\right]$ and $\left[\, n \text{ is even} \overset{\text{ER 3.2.1}}{\Longleftrightarrow} n^3 \text{ is even} \,\right]$.

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7. Prove Theorem 7.

Theorem 7. Let a and b be integers with $a \neq 0$. If a does not divide b, then the equation

$$ax^3 + bx + (b + a) \;=\; 0 \tag{7.1}$$

does not have a solution that is a natural number.

hint. Proof by contrapositive. So assume there exists $n \in \mathbb{N}$ that is a solution to equation in (7.1), i.e., $an^3 + bn + (b + a) = 0$. Note $ax^3 + bx + (b + a) = 0$ when $x = -1$ so $x - (-1) \overset{\text{i.e.}}{=} x + 1$ is a factor of $x^3 + bx + (b + a)$. Now use long division of polynomials to get $n^3 + 1 \;=\; (n + 1)\,(n^2 - n + 1)$.

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def. A **Pythagorean triple** is a 3-tuple (a, b, c) such that $a, b, c \in \mathbb{N}$ with $a < b < c$ and $a^2 + b^2 = c^2$.

8. Are the following propositions true or false? Justify your conclusions.

8.1. If (a, b, c) form a Pythagorean triple, then 3 divides a or 3 divides b.

8.2. If (a, b, c) form a Pythagorean triple, then 5 divides a or 5 divides b or 5 divides c.

hint. Both are true. Helpful Facts. Lemma 8.1: if $a \in \mathbb{Z}$, then $[a \not\equiv 0 \pmod 3] \Leftrightarrow [a^2 \equiv 1 \pmod 3]$. Pf: $\boxed{\Rightarrow}$ is proof by cases. $\boxed{\Leftarrow}$ is proof by contrapositive. Look over your class notes. Lemma 8.2. If $c \in \mathbb{Z}$, then $c^2 \equiv 0 \pmod 3$ or $c^2 \equiv 1 \pmod 3$. Pf: use Lemma 8.1. Proposition 3.33. If $a \in \mathbb{Z}$ and $a \not\equiv 0 \pmod 5$, then $a^2 \equiv 1 \pmod 5$ or $a^2 \equiv 4 \pmod 5$. Pf: by cases, book §3.5 (p151).

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9. More on Pythagorean triples.

9.1. Prove that there exists a Pythagorean triple (a, b, c) where $a = 5$ and b and c are consecutive natural numbers.

9.2. Prove that there exists a Pythagorean triple (a, b, c) where $a = 7$ and b and c are consecutive natural numbers.

9.3. Prove that there exists a Pythagorean triple (a, b, c) where a is odd and b and c are consecutive natural numbers.

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11. Definition. Two prime numbers that differ by 2 are called **twin primes**.

Conjecture 11. If p and q are twin primes other than 3 and 5, then $pq + 1$ is a perfect square and 36 divides $pq + 1$.

Is Conjecture 11 true or false? Justify your answer.

hint. Let p and q be twin primes. Without loss of generality, $p < q$. So $q = p + 2$. Now express $pq + 1$ in terms of only p (i.e., get rid of q). Note 36 divides a perfect square k^2 if and only if 6 divides k. Cases with p and mod 6.

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12. Are the following statements true or false? Justify your conclusions.

12.1. If a and b are integers, then $(a + b)^2 \equiv (a^2 + b^2) \pmod 2$.

12.2. If a and b are integers, then $(a + b)^3 \equiv (a^3 + b^3) \pmod 3$.

12.3. If a and b are integers, then $(a + b)^4 \equiv (a^4 + b^4) \pmod 4$.

12.4. If a and b are integers, then $(a + b)^5 \equiv (a^5 + b^5) \pmod 5$.

hint. Parts 12.1, 12.2, 12.4 are true. Part 12.3 is false. Algebra gives: $(a + b)^2 - (a^2 + b^2) = (a^2 + 2ab + b^2) - (a^2 + b^2) = 2ab$. For the higher powers, if needed, review (the linked) Pascal's Triangle.
