Theorem 1. An integer n is even if and only if n^2 is an even. TS§3.2 Thm3.10 p108
Symbolically written: $(\forall n \in \mathbb{Z})$ [n is even $\Leftrightarrow$ n^2 is even]

Proof by Previous Shown Results. Let $n \in \mathbb{Z}$. We will show that n is even if and only if n^2 is even by showing the implication holds in both directions.

$\boxed{\Rightarrow}$. First we will show that if n is even then n^2 is even. Let n be even. Since n is even, $n^2 = n \cdot n$, and the product of an even integer and any integer is even [cf. Lemma PEA], n^2 is even. We have just shown if n is even then n^2 is even.

$\boxed{\Leftarrow}$. Next we will show that if n^2 is even then n is even. Let n^2 by even. By Theorem S from §2.1, which states that if n^2 is even for an integer n then n is even, we get that n is even. We have just shown if n^2 is even then n is even.

By showing the implication holds in both directions in Theorem 1, we have show that an integer n is even if and only if n^2 is even. □

Proof using definitions. Let $n \in \mathbb{Z}$. We will show that n is even if and only if n^2 is even by showing the implication holds in both directions.

$\boxed{\Rightarrow}$. First we will show that if n is even then n^2 is even. Let n be even. Since n is even, by definition of an even integer, there is $s \in \mathbb{Z}$ such that

$$n = 2s. \tag{1}$$

By (1)

$$n^2 \;=\; (2s)\,(2s) \;=\; 2\left(2s^2\right) \;=\; 2t \tag{2}$$

where $t = 2s^2$. By the closure properties of the integers, $t \in \mathbb{Z}$. Thus (2) gives that n^2 is even by definition of even. We have just shown if n is even then n^2 is even.

$\boxed{\Leftarrow}$. Next we will show that if n^2 is even then n is even $\langle$think: $R \implies S\rangle$ by using the contrapositive, which is $\langle \sim S \implies \sim R\rangle$, if n is odd then n^2 is odd. So now let n be odd and we will show that n^2 is odd. Since n is odd, by definition of an odd integer, there is $k \in \mathbb{Z}$ such that

$$n = 2k + 1. \tag{3}$$

By (3) and then algebra

$$\begin{aligned} n^2 &= (2k+1)\,(2k+1) \\ &= 4k^2 + 4k + 1 \\ &= 2\left(2k^2 + 2k\right) + 1 \\ &= 2j + 1 \end{aligned} \tag{4}$$

where $j = 2k^2 + 2k$. By the closure properties of the integers, $j \in \mathbb{Z}$. Thus (4) gives that n^2 is odd by definition of odd. We have just shown that if n is an odd integers then n^2 is an odd integer. Thus, by the contrapositive, if n^2 is an even integer then n is an even integer.

By showing the implication holds in both directions in Theorem 1, we have show that an integer n is even if and only if n^2 is even. □

Preview:

We now have shown:

- **Thm. 3.10**. $(\forall n \in \mathbb{Z})$ [n is even $\Leftrightarrow$ n^2 is even] p108

and since $[R \Leftrightarrow S] \equiv [\sim R \Leftrightarrow \sim S]$ and also for an integer *not being even* is the same as *being odd*

- **Cor. 3.10**. $(\forall n \in \mathbb{Z})$ [n is odd $\Leftrightarrow$ n^2 is odd] . NotInBk

Homework will be to show:

- **ER. 3.2.1c**. $(\forall n \in \mathbb{Z})$ [n is even $\Leftrightarrow$ n^3 is even]
- **ER. 3.2.1d**. $(\forall n \in \mathbb{Z})$ [n is odd $\Leftrightarrow$ n^3 is odd] .

ERhint. As usual, on a specific ER (unless otherwise indicated) you may use any previous: ER or result from class.

Recall Theorem 1

Theorem 1. $(\forall n \in \mathbb{Z})$ [n is even $\Leftrightarrow$ n^2 is even] . TS§3.2 Thm3.10 p108

In our *proof by Previously Shown Results*,

- the forward direction $\boxed{\Rightarrow}$ used Lemma PEA,
- while the backwards direction $\boxed{\Leftarrow}$ by §2.1's Theorem S.

We could not somehow *combine both directions into one argument* since the two directions used a different Previous Shown Result.

In our *proof using definitions*,

- the forward direction $\boxed{\Rightarrow}$ used a direct proof,
- while the backwards direction $\boxed{\Leftarrow}$ by used the contrapositive.

We could not somehow *combine both directions into one argument* since the two directions used different methods of proof.

Compare the situation in Thm. 1 to the situation in Thm. 2.

Theorem 2

Theorem 2. Let $n \in \mathbb{N}$ and $a \in \mathbb{Z}$. It holds that $a \equiv 0 \pmod{n}$ if and only if n divides a. TS§3.1 ER3.1.7ab p97
Symbolically written: $(\forall n \in \mathbb{N})\ (\forall a \in \mathbb{Z})\ [\ a \equiv 0 \pmod{n} \ \Leftrightarrow\ n \mid a\ ]$

Thinking Land for Theorem 2

Let $n \in \mathbb{N}$ and $a \in \mathbb{Z}$. Note: $a \equiv 0 \pmod{n} \xLeftrightarrow[\text{module congruence}]{\text{by def. of}} n|\,(a-0) \xLeftrightarrow{\text{by algebra}} n|a.$

Proof of Theorem 2

Proof of Theorem 2. Let $n \in \mathbb{N}$ and $a \in \mathbb{Z}$. We will show that $a \equiv 0 \pmod{n}$ if and only if $n \mid a$.

Note the following are equivalent. The expression

$$a \equiv 0 \pmod{n}$$

and, by definiton of congruence,

$$n \mid (a-0)$$

and by algebra

$$n \mid a.$$

Thus, we have shown that $a \equiv 0 \pmod{n}$ is equalent to $n \mid a$.

This completes the proof. □