1. Definitions/Key Idea

1.1. A **statement** is a declarative sentence that is either true or false but not both; thus, a statement has exactly one **truth value**: true (T) or false (F). §1.1 p1
We often represent a statement by a letter (such as P), called a **statement variable**, similar to the way we represent a number by a variable (such as x).

▷. An example illustrating definitions to come. The compound statement $(P \Rightarrow Q) \vee (R \Leftrightarrow (\sim Q))$ has 3 atoms (namely: P, Q, R) and 4 connectives (namely: $\Rightarrow$, $\vee$, $\Leftrightarrow$, $\sim$).

1.2. A **logical operator** (or **connective**) on statement(s) is a word or combinations of words (e.g.: impplies, and, or, if-then) that combines one or more statements to make a new statement. §2.1 p33

1.3. An **atomic statement** (or **atom**) is a statement satisfying that no part of it is itself a statement. (e.g., P) ∄

1.4. A **compound statement** is a statement that contains one or more connectives. §2.1 p33
A compound statement can be decomposed into its atom(s) and connective(s).

1.5. A **truth table** of a statement exhibits the truth values (TorF) of the statement for each possible combination of truth values for its atoms. §1.1 p6

1.6. A **tautology** is a statement that is true for each assignment of truth values to its atom(s). §2.1 p40

1.7. A **contradiction** is a statement that is false for each assignment of truth values to its atom(s). §2.1 p40

1.8. Two statements $\widetilde{P}$ and $\widetilde{Q}$ are **(logically) equivalent** provided they have the same truth value for each possible combinations of truth values for all the atoms appearing in $\widetilde{P}$ and $\widetilde{Q}$. §2.2 p43
We denote $\widetilde{P}$ is (logically) equivalent to $\widetilde{Q}$ by: $\widetilde{P} \equiv \widetilde{Q}$.
Note, $\equiv$ is used between statements while $=$ is used between numbers.

2. Connective Symbols and Truth Tables

2.1.

	negation
P	∼ P
T	F
F	T

		conjunction and	disjunction or	conditional (implication/if-then)	biconditional (if and only if)
P	Q	P∧ Q	P∨Q	P⇒Q	P⇔Q
T	T	**T**	T	T	T
T	F	F	T	**F**	F
F	T	F	T	T	F
F	F	F	**F**	T	T

2.2. The truth table of a compound statement consisting of n atoms has 2^n lines.
This exhausts all possible truth values of the atoms.

On written work, follow the book's (as done above) pattern of the last listed atomic statement column has alternating T's and F's.

2.3. As in algebra, we give connectives a priority ordering to resolves ambiguities when parentheses are omitted. ∄
The **priority/precedence of the connectives** is:

$$\sim \text{(high, so do first)} , \wedge \ , \vee \ , \Rightarrow \ , \Leftrightarrow \text{(low, so do last)} .$$

Ex. $P \implies \sim Q \vee R \Leftrightarrow S$ is an abbreviation for $(P \implies [(\sim Q) \vee R]) \Leftrightarrow S$.

2.4. Use $P \Rightarrow Q$ to translate: (Good check if you have grasp of 2.4 is the book's ER 2.1.13 p. 42).) §2.1 p37

(1) If P, then Q

(2) P implies Q

(3) P is sufficient for Q

(4) P only if Q

(5) P only when Q

(6) Q is necessary for P

(7) Q if P

(8) Q when P (also Q whenever P)

Rmk on (4). What sounds correct to your ear?

- ◦ $x > 0$ only if $x > 17$.
- ◦ $x > 17$ only if $x > 0$.

2.5. Use $P \Leftrightarrow Q$ to translate: §2.1 p39

(1) P is equivalent to Q

(2) P if and only if Q (See above $P \Rightarrow Q$'s (4) and (7))

(3) P if but only if Q

(4) P precisely when Q

(5) P is necessary and sufficient for Q (See above $P \Rightarrow Q$'s, (3) and (6))

Ex. Construct a truth table (all in 1 big truth table) for the following 4 compound statements.

(a) $P \Rightarrow Q$ (b) $Q \Rightarrow P$ (d) $(\sim Q) \Rightarrow (\sim P)$

(e) $[\ P \Rightarrow Q\ ] \ \Leftrightarrow\ [\ (\sim Q) \Rightarrow (\sim P)\ ]$. ⟨Recall on truth tables, it is easiest/fastest to do column-wise⟩

				(a)	(b)	(d)	(e)
P	Q	$\sim Q$	$\sim P$	$P \Rightarrow Q$	$Q \Rightarrow P$	$(\sim Q) \Rightarrow (\sim P)$	$[\ P \Rightarrow Q\ ] \ \Leftrightarrow\ [\ (\sim Q) \Rightarrow (\sim P)\ ]$
T	T						
T	F						
F	T						
F	F						

. . . .

1. Which columns are the same? ________________________________

2. Which of the 4 statements is a tautology? ________________________________

Justify:

3. Which of the 4 statements is a contradiction? ________________________________

Justify:

4a. Is $[\ P \Rightarrow Q\ ] \ \equiv\ [\ (\sim Q) \Rightarrow (\sim P)\ ]$? (circle one) Yes / No .

Justify:

4b. **Def**. The **contrapositive** of the conditional statement $P \Rightarrow Q$ is the conditional statement $(\sim Q) \Rightarrow (\sim P)$.

4c. Is a conditional statement logically equivalent to it's contrapositive? (circle one) Yes / No .

Justify:

5a. Is $[\ P \Rightarrow Q\ ] \ \equiv\ [\ Q \Rightarrow P\ ]$? (circle one) Yes / No .

Justify:

5b. **Def**. The **converse** of the conditional statement $P \Rightarrow Q$ is the conditional statement $Q \Rightarrow P$.

5c. Is a conditional statement logically equivalent to it's converse? (circle one) Yes / No .

Justify:

. . . .

★. Lesson Learned.

We want to show $P \Rightarrow Q$ but it is kinda not looking to promising (since it is too hard to show).

What can we do (and not do)?

Review

Def. Two statements $\widetilde{P}$ and $\widetilde{Q}$ are **(logically) equivalent** provided they have the same truth value for each possible combinations of truth values for all the atoms appearing in $\widetilde{P}$ and $\widetilde{Q}$. §2.2 p43
We denote $\widetilde{P}$ is (logically) equivalent to $\widetilde{Q}$ (i.e., $\widetilde{P}$ and $\widetilde{Q}$ are (logically) equivalent) by: $\widetilde{P} \equiv \widetilde{Q}$.
Note, $\equiv$ is used between statements while $=$ is used between numbers.

Def.
- The **converse** of the conditional statement $P \Rightarrow Q$ is the conditional statement $Q \Rightarrow P$. p44
- The **contrapositive** of the conditional statement $P \Rightarrow Q$ is the condition statement $(\sim Q) \Rightarrow (\sim P)$.
- **Rmk**. We have already seen that: $[P \Rightarrow Q] \not\equiv [Q \Rightarrow P]$ but $[\,P \Rightarrow Q] \equiv [(\sim Q) \Rightarrow (\sim P)\,]$.

Def. A **negation** (also called **denial**) of a statement P is $\sim P$. p33

Recall. The priority of connectives is: $\sim$ (high, so do first) , $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$ (low, so do last).
So $\sim P \vee \sim Q$ is an abbreviation for $(\sim P) \vee (\sim Q)$.

Important Logical Equivalencies

Theoremm 2.8. Let P, Q, and R be statements. Thm 2.8 §2.2 p48

Double Negation:

$$[\,\sim (\sim P)\,] \;\equiv\; P\,. \tag{1}$$

Biconditional Statement:

$$[\,P \Leftrightarrow Q\,] \;\equiv\; [\,(P \Rightarrow Q) \wedge (Q \Rightarrow P)\,]\;. \tag{2}$$

De Morgans Laws:

$$[\sim (P \wedge Q)\,] \;\equiv\; [\,(\sim P) \vee (\sim Q)\,] \tag{3}$$
$$[\,\sim (P \vee Q)\,] \;\equiv\; [\,(\sim P) \wedge (\sim Q)\,]\;. \tag{4}$$

Distributive Laws:

$$[\,P \vee (Q \wedge R)\,] \;\equiv\; [\,(P \vee Q) \wedge (P \vee R)\,] \tag{5}$$
$$[\,P \wedge (Q \vee R)\,] \;\equiv\; [\,(P \wedge Q) \vee (P \wedge R)\,]\;. \tag{6}$$

Conditional Statements:

$$[\,P \Rightarrow Q\,] \;\equiv\; [\,(\sim Q) \Rightarrow (\sim P)\,] \qquad \text{(contrapositive)} \tag{7}$$
$$[\,P \Rightarrow Q\,] \;\equiv\; [\,(\sim P) \vee Q\,] \qquad \text{(how do you keep a promise?)} \tag{8}$$
$$[\,\sim (P \Rightarrow Q)\,] \;\equiv\; [\,P \wedge (\sim Q)\,] \qquad \text{(how do you break a promise?)} \tag{9}$$
$$[\,\sim (P \wedge Q)\,] \;\equiv\; [\,P \Rightarrow (\sim Q)\,]\;. \qquad \text{(not in book)} \tag{10}$$

Conditionals with Disjunctions:

$$[\,(P \vee Q) \Rightarrow R\,] \;\equiv\; [\,(P \Rightarrow R) \wedge (Q \Rightarrow R)\,] \tag{11}$$
$$[P \Rightarrow (Q \vee R)\,] \;\equiv\; [\,(P \wedge (\sim Q)) \Rightarrow R\,]\;. \tag{12}$$

Defs:. p54

1. A **variable** is a symbol representing an arbitrary (i.e., unspecified, generic) object that can be chosen from a given set U.
2. The set U is called the **universal set for the variable**. So the *universal set for the variable* is the set of specified objects from which objects may be chosen to substitute for the variable.
3. A **constant** is a specific member of the universal

Ex1. So in $x \in \mathbb{R}$, the x is the variable and the universe is $\mathbb{R}$. A constant would be 17.

Defs:. p56/58

4. An **open sentence** is a sentence $P(x_1, x_2, \ldots, x_n)$ involving variables $x_1, x_2, \ldots, x_n$ with the property that when specific values from the universal set are assigned to $x_1, x_2, \ldots, x_n$, the result is a statement (i.e., a declaritive sentence that is either true or false, but not both).
5. BTW: in other classes, you might hear an open sentence called a **predicate** or a **propositional function**.
6. The **truth set of an open sentence with one variable** is the collection of objects in the universal set that can be substituted for the variable to make the open sentence a true statement.

Ex2. An example of an open sentence $P(x)$ is $x^2 = 9$, with the universe being $\mathbb{R}$.
The truth set for $P(x)$ is $\{3, -3\}$.

►. Set builder notation takes the form

$$\{x \in U \colon P(x)\}$$

where x is the variable, U is the universal set, and $P(x)$ is the rule/restriction/property that the variable x must satify to be in the set. In Example 2 above, we considered the set

$$\left\{x \in \mathbb{R} \colon x^2 = 9\right\}.$$

Note

$$\underbrace{\left\{x \in \mathbb{R} \colon x^2 = 9\right\}}_{\text{set builder notation}} = \underbrace{\{3, -3\}}_{\text{roster method}}.$$

The roster method just lists the set's elements between curly braces.

Ex3. Some Set Notation for the set $A \overset{\text{def}}{:=} \{5, 9, 13, 17, 21, 25 \ldots\}$

$$\underbrace{\{5, 9, 13, 17, 21, 25 \ldots\}}_{\text{roster method}} \overset{\text{increase}}{\underset{\text{by 4 so}}{=}} \left\{ \overbrace{4(0)+5}^{=5}, \overbrace{4(1)+5}^{=9}, \overbrace{4(2)+5}^{=13}, \overbrace{4(3)+5}^{=17}, \overbrace{4(4)+5}^{=21}, \overbrace{4(5)+5}^{=25}, \ldots \right\}$$

$$= \underbrace{\{4n+5 \in \mathbb{N} \colon n \in \{0\} \cup \mathbb{N}\}}_{\text{set notation but not set builder notation}}$$

$$= \underbrace{\left\{ x \in \mathbb{N} \colon \overbrace{x = 4n+5 \ \text{ for some } \ n \in \{0\} \cup \mathbb{N}}^{P(x)} \right\}}_{\text{set builder notation}}$$

and if we want to replace $\{0\} \cup \mathbb{N}$ with just $\mathbb{N}$, then what adjustments needed? well ...

$$= \left\{ \overbrace{4(1)+1}^{=5}, \overbrace{4(2)+1}^{=9}, \overbrace{4(3)+1}^{=13}, \overbrace{4(4)+1}^{=17}, \overbrace{4(5)+1}^{=21}, \overbrace{4(6)+1}^{=25}, \ldots \right\}$$

$$= \underbrace{\{4n+1 \in \mathbb{N} \colon n \in \mathbb{N}\}}_{\text{set notation but not set builder notation}} \quad \langle \text{think DA: divide a number in } A \text{ by 4 and will get remainder 1} \rangle$$

$$= \underbrace{\left\{ y \in \mathbb{N} \colon \overbrace{y = 4n+1 \ \text{ for some } \ n \in \mathbb{N}}^{P(x)} \right\}}_{\text{set builder notation}}.$$

Ex4. In class.

Recall Some Set Theory/Notation

Defs. Let A and B be subsets of some universal set U.

1. An **empty set** is a set that contains no elememts. We denote an empty set by $\emptyset$. §2.3 p60

   ⟨The symbol $\emptyset$ is the last letter in the Danish-Norwegian alphabet⟩. Latex: `\emptyset`

2. The subset $A \setminus B$ of U is defined by §5.1 p216

$$A \setminus B \overset{\text{def}}{=} \{x \in U \colon x \in A \quad \text{but} \quad x \notin B\}.$$

   The subset $A \setminus B$ is called a **set difference** and is read: A set minus B or A set take away B.

3. The set A is a **subset** B provided that each element of A is an element of B. §2.3 p55

   If A is a subset of B, then we write $A \subseteq B$ and can say any of the following:

   - A is a **subset** of B
   - A is **contained** in B
   - B **contains** A
   - B is a **superset** of A.

   When A is not a subset of B, we write $A \not\subseteq B$.

4. The sets A and B are **equal** when they have precisely the same elements. §2.3 p55

   If A and B are equal, then we write $A = B$. If A and B are not equal, then we write $A \neq B$.

Helpful in Proofs

Rmk. $[A \subseteq B] \overset{\text{by def.}}{\iff} [\ x \in A \implies x \in B\ ]$

$[B \subseteq A] \overset{\text{by def.}}{\iff} [\ x \in B \implies x \in A\ ]$

$[A = B] \overset{\text{by def.}}{\iff} [\ x \in A \iff x \in B\ ]$ ... so we get ... $[A = B] \overset{\text{so get}}{\iff} [\ (A \subseteq B) \wedge (B \subseteq A)\ ]$

Ex5. In class.

Four definitions (from number theory) used in the homework exercises.

Def. A $p \in \mathbb{N}$ is **prime** provided $p \neq 1$ and the only natural numbers that are factors of p are: 1 and p. p78

Def. A $c \in \mathbb{N}$ is **composite** provided $c \neq 1$ and c is not a prime number. p78

▷. The number 1 is neither prime nor composite. In Math 546 you will learn 1 is a unit.

Def. An integer n is a **multiple of 3** provided: $(\exists k \in \mathbb{Z})\ [n = 3k]$. p71

Def. A natural number n is a **perfect square** provided: $(\exists k \in \mathbb{N})\ [n = k^2]$. p70

Def. The phrase *for all* (or its equivalents) is a **universal quantifier** and is denoted by $\forall$. p63
The phrase *there exists* (or its equivalents) is an **existential quantifier** and is denoted by $\exists$.

Rmk. The symbol $\exists!$ reads *there exists a unique*. NotInBk

⟨So $\exists!$ means there exitsts one and only one. Compare to $\exists$, which means there exists at least one.⟩

Rmk. Priority/precedence when parentheses are excluded: $\forall$ and $\exists$ and $\exists!$ have equal priority and come after the logical connective symbols: $\sim$ (high, so do first) , $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$ (low, so do last). NotInBk

Statements with one quantifer

►. Example/Terminology of statement with one qualifier. p64

$$\underbrace{\overbrace{(\ \forall x \in U\ )}^{\text{quantifies the variable } x} \qquad \overbrace{[\ P(x)\ ]}^{\text{open sentence in the variable } x}}_{\text{a statement}}$$

Let $P(x)$ be an open sentence of the variable x from the universe U.		
a statement involving	**often has the forms**	**the statement is true provided**
universal quantifier $(\forall x \in U)\ [P(x)]$	For all $x \in U$, $P(x)$. For every $x \in U$, $P(x)$. For each $x \in U$, $P(x)$.	$P(x)$ is true for all $x \in U$.
existential quantifier $(\exists x \in U)\ [P(x)]$	There exists an $x \in U$ such that $P(x)$. There is an $x \in U$ such that $P(x)$.	$P(x)$ is true for at least one $x \in U$.
$(\exists! x \in U)\ [P(x)]$	There exists a unique $x \in U$ such that $P(x)$.	$P(x)$ is true for precisely one (and only one) $x \in U$.

?. Where does the pharse such that appear in the above chart?

Def. A **counterexample** to a statement is an example that shows the statement is false. p69
So a counterexample to a statement of the form $(\forall x \in U)\ [P(x)]$ is an example that shows $(\forall x \in U)\ [P(x)]$ is false; more specifically, an example/specific-element/constant $c \in U$ for which $P(c)$ is false.
<So to show a statement is false, we can find a counterexample to the statement.
To show a statement is true, we prove the statement.>

Ex0. Review our Symbolically Write Guidelines.

Ex1.O. Do Example 1 part O for Ex. 1.1–1.3. ⟨The O is for original (statement)⟩. ⟨see last page⟩

Thm. Negations of Quantified Statements. For an open sentence $P(x)$, Thm2.16 p67

$$\sim \{\ (\forall x \in U)\ [P(x)]\ \} \quad \equiv \quad (\exists x \in U)\ [\sim P(x)]$$
$$\sim \{\ (\exists x \in U)\ [P(x)]\ \} \quad \equiv \quad (\forall x \in U)\ [\sim P(x)]$$

Ex1.N. Do Example 1 part N for Ex. 1.1–1.3. ⟨The N is for negation (of the original statement)⟩. ⟨see last page⟩

Statements with 2 Like Quantifiers
two ∀ or two ∃

▸. As we saw in Example 1, we can interchange two ∀ in a row. More specifically, let $R(x,y)$ for an open sentence in the variables x in universe U_1 and y in unviverse U_2. Then

$$[\,(\forall x \in U_1)\,(\forall y \in U_2)\,[\,R(x,y)\,]\,] \equiv [\,(\forall y \in U_2)\,(\forall x \in U_1)\,[\,R(x,y)\,]\,]. \qquad (1)$$

Taking the negation of both sides of (1) gives (the un-useful negations)

$$\sim [\,(\forall x \in U_1)\,(\forall y \in U_2)\,[\,R(x,y)\,]\,] \equiv \sim [\,(\forall y \in U_2)\,(\forall x \in U_1)\,[\,R(x,y)\,]\,]. \qquad (2)$$

Cleaning up the un-useful negations in (2) gives

$$[\,(\exists x \in U_1)\,(\exists y \in U_2)\,[\,\sim R(x,y)\,]\,] \equiv [\,(\exists y \in U_2)\,(\exists x \in U_1)\,[\,\sim R(x,y)\,]\,]. \qquad (3)$$

Denote the open sentence $\sim R(x,y)$ by the open sentence $S(x,y)$ to see that (3) gives

$$[\,(\exists x \in U_1)\,(\exists y \in U_2)\,[\,S(x,y)\,]\,] \equiv [\,(\exists y \in U_2)\,(\exists x \in U_1)\,[\,S(x,y)\,]\,]. \qquad (4)$$

So we can also interchange two ∃ in a row, as seen by (4)

⚠. **Lesson Learned**. We can interchange/switch the order of 2 like/same quantifiers in a row !!!!

?. **Question**. What if we have two *mixed* quantifiers, i.e. one ∃ and one ∀?
Can we still interchange the order of the quantifiers? We will find out in the next example.

Statements with 2 Mixed Quantifiers
one ∀ and one ∃

	Symbolic Form	**English Form**
Statement	$(\exists x \in \mathbb{Z})\ (\forall y \in \mathbb{Z})\ [x + y = 0]$	There exists an integer x such that for each integer y, we have $x + y = 0$.
Negation	$(\forall x \in \mathbb{Z})\ (\exists y \in \mathbb{Z})\ [x + y \neq 0]$	For each integer x, there exists an integer y such that $x + y \neq 0$.

	Symbolic Form	**English Form**
Statement	$(\forall x \in \mathbb{Z})\ (\exists y \in \mathbb{Z})\ [x + y = 0]$	For each integer x, there is an integer y such that $x + y = 0$.
Negation	$(\exists x \in \mathbb{Z})\ (\forall y \in \mathbb{Z})\ [x + y \neq 0]$	There is an integer x such that for each integer y, we have $x + y \neq 0$.

Ex 2a. $(\forall x \in \mathbb{Z})\ (\exists y \in \mathbb{Z})\ [x + y = 0]$ circle one: true or false

Ex 2b. $(\exists y \in \mathbb{Z})\ (\forall x \in \mathbb{Z})\ [x + y = 0]$ circle one: true or false

Ex 2c. If we interchange the order of two mixed quantifiers, do we (always) get the same truth value? ____

Ex 1. Read the Symbolically Write Guidelines, which are posted on the course Handout page.
Below are variants of statements from previous Exercises. For each Exercise:

O. Symbolically write (using quantifiers) the original statement.
Then indicate whether the original statement is true or false (no justification needed).

N. Symbolically write (using quantifiers) a useful negation of the original statement. Box your answer.
Then indicate whether the negation of the original statement is true or false (no justification needed).

1.1. If m is an odd integer, then $5m + 6$ is an even integer. ≈ER 1.2.4b p27

1.2. If m and n are odd integers, then $mn + 7$ is an even integer.
I.e., The sum of 7 and the product of 2 odd integers is an even integer. ER 1.2.4c p27

1.3. If a, b, and c are natural numbers, then $ab^2 + a^3c^4 + a^{100}b^{100}c^{100}$ is an odd natural number. ≈ER 1.2.7a p28