Four definitions (from number theory) used in the homework exercises.

Def. A $p \in \mathbb{N}$ is **prime** provided $p \neq 1$ and the only natural numbers that are factors of p are: 1 and p. p78

Def. A $c \in \mathbb{N}$ is **composite** provided $c \neq 1$ and c is not a prime number. p78

▷. The number 1 is neither prime nor composite. In Math 546 you will learn 1 is a unit.

Def. An integer n is a **multiple of 3** provided: $(\exists k \in \mathbb{Z})\ [n = 3k]$. p71

Def. A natural number n is a **perfect square** provided: $(\exists k \in \mathbb{N})\ [n = k^2]$. p70

Def. The phrase *for all* (or its equivalents) is a **universal quantifier** and is denoted by $\forall$. p63
The phrase *there exists* (or its equivalents) is an **existential quantifier** and is denoted by $\exists$.

Rmk. The symbol $\exists!$ reads *there exists a unique*. NotInBk

⟨So $\exists!$ means there exitsts one and only one. Compare to $\exists$, which means there exists at least one.⟩

Rmk. Priority/precedence when parentheses are excluded: $\forall$ and $\exists$ and $\exists!$ have equal priority and come after the logical connective symbols: $\sim$ (high, so do first) , $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$ (low, so do last) . NotInBk

Statements with one quantifer

►. Example/Terminology of statement with one qualifier. p64

$$\underbrace{\overbrace{(\ \forall x \in U\ )}^{\text{quantifies the variable } x}\quad \overbrace{[\ P(x)\ ]}^{\text{open sentence in the variable } x}}_{\text{a statement}}$$

Let $P(x)$ be an open sentence of the variable x from the universe U.		
a statement involving	**often has the forms**	**the statement is true provided**
universal quantifier $(\forall x \in U)\ [P(x)]$	For all $x \in U$, $P(x)$. For every $x \in U$, $P(x)$. For each $x \in U$, $P(x)$.	$P(x)$ is true for all $x \in U$.
existential quantifier $(\exists x \in U)\ [P(x)]$	There exists an $x \in U$ such that $P(x)$. There is an $x \in U$ such that $P(x)$.	$P(x)$ is true for at least one $x \in U$.
$(\exists! x \in U)\ [P(x)]$	There exists a unique $x \in U$ such that $P(x)$.	$P(x)$ is true for precisely one (and only one) $x \in U$.

?. Where does the pharse such that appear in the above chart?

Def. A **counterexample** to a statement is an example that shows the statement is false. p69
So a counterexample to a statement of the form $(\forall x \in U)\ [P(x)]$ is an example that shows $(\forall x \in U)\ [P(x)]$ is false; more specifically, an example/specific-element/constant $c \in U$ for which $P(c)$ is false.
<So to show a statement is false, we can find a counterexample to the statement.
To show a statement is true, we prove the statement.>

Ex0. Review our Symbolically Write Guidelines. ⟨either click the blue link or see next page⟩

Ex1.O. Do Example 1 part O. ⟨The O is for original (statement)⟩. ⟨see last page⟩

Thm. Negations of Quantified Statements. For an open sentence $P(x)$, Thm2.16 p67

$$\sim \{\ (\forall x \in U)\ [P(x)]\ \} \quad \equiv \quad (\exists x \in U)\ [\sim P(x)]$$
$$\sim \{\ (\exists x \in U)\ [P(x)]\ \} \quad \equiv \quad (\forall x \in U)\ [\sim P(x)]$$

Ex1.N. Do Example 1 part N. ⟨The N is for negation (of the original statement)⟩. ⟨see last page⟩

Henceforth, when asked to *symbolically write* a statement follow the below (unless otherwise stated).

(1) If a statement is a quantified open sentence, then use needed quantifier(s) (e.g.: $\forall$, $\exists$, $\exists!$).
Recall
- $\forall$ reads *for alll*
- $\exists$ reads *there exists*
- $\exists!$ reads *there exists a unique.*

(2) Use logical connectives symbols (e.g.: $\sim$, $\wedge$, $\vee$, $\implies$, $\iff$) instead of the English words.

(3) Within an open sentence, you can use English words that are not logical connectives words.
E.g., within your open sentence, one can write: "x is even".
Beware: "x and y are odd" should be expressed as "x is odd $\wedge$ y is odd".

(4) Within an open sentence, you can use math symbols that are not logical connectives.
So you may use, e.g.: $x < \sqrt{2}$, $x = y$, $x + y = 17$, $a|b$, $a \equiv b \pmod{n}$, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

(5) Symbolically write the statement as it is stated (rather than something logically equivalent).
For example, the statement

$$\text{if a real number is larger than 3, then its square is larger than 9} \tag{1}$$

can be symbolically written as

$$(\forall x \in \mathbb{R}) \left[x > 3 \implies x^2 > 9 \right] . \tag{yes}$$

The statement in (1) is formulated as in (yes) so symbolically write (1) as (yes).

Do NOT symbolically write the statement in (1) as

$$(\forall x \in \mathbb{R}) \left[x^2 \leq 9 \implies x \leq 3 \right] \tag{no}$$

since (no) is not as (1) is formulated. Do note ⟨will help you later⟩ that the statement in (yes) is logically equivanlent to the statement in (no) ⟨since $[P \implies Q] \equiv [(\sim Q) \implies (\sim P)]$, think contrapositive⟩; thus, if you need to prove the statement in (1), then you can prove (yes) or (no) ⟨choice is yours when proving⟩.

Statements with 2 Like Quantifiers
two $\forall$ or two $\exists$

►. As we saw in Example 1, we can interchange two $\forall$ in a row. More specifically, let $R(x,y)$ for an open sentence in the variables x in universe U_1 and y in unviverse U_2. Then

$$[\ (\forall x \in U_1)\ (\forall y \in U_2)\ [\ R(x,y)\ ]\ ] \ \equiv\ [\ (\forall y \in U_2)\ (\forall x \in U_1)\ [\ R(x,y)\ ]\ ]\,. \tag{1}$$

Taking the negation of both sides of (1) gives (the un-useful negations)

$$\sim [\ (\forall x \in U_1)\ (\forall y \in U_2)\ [\ R(x,y)\ ]\ ] \ \equiv \sim [\ (\forall y \in U_2)\ (\forall x \in U_1)\ [\ R(x,y)\ ]\ ]\,. \tag{2}$$

Cleaning up the un-useful negations in (2) gives

$$[\ (\exists x \in U_1)\ (\exists y \in U_2)\ [\ \sim R(x,y)\ ]\ ] \ \equiv\ [\ (\exists y \in U_2)\ (\exists x \in U_1)\ [\ \sim R(x,y)\ ]\ ]\,. \tag{3}$$

Denote the open sentence $\sim R(x,y)$ by the open sentence $S(x,y)$ to see that (3) gives

$$[\ (\exists x \in U_1)\ (\exists y \in U_2)\ [\ S(x,y)\ ]\ ] \ \equiv\ [\ (\exists y \in U_2)\ (\exists x \in U_1)\ [\ S(x,y)\ ]\ ]\,. \tag{4}$$

So we can also interchange two $\exists$ in a row, as seen by (4)

⚠. **Lesson Learned**. We can interchange/switch the order of 2 like/same quantifiers in a row !!!!

?. **Question**. What if we have two *mixed* quantifiers, i.e. one $\exists$ and one $\forall$?
Can we still interchange the order of the quantifiers? We will find out in the next example.

Statements with 2 Mixed Quantifiers
one $\forall$ and one $\exists$

	Symbolic Form	**English Form**
Statement	$(\exists x \in \mathbb{Z})\ (\forall y \in \mathbb{Z})\ [x+y=0]$	There exists an integer x such that for each integer y, we have $x+y=0$.
Negation	$(\forall x \in \mathbb{Z})\ (\exists y \in \mathbb{Z})\ [x+y\neq 0]$	For each integer x, there exists an integer y such that $x+y\neq 0$.

	Symbolic Form	**English Form**
Statement	$(\forall x \in \mathbb{Z})\ (\exists y \in \mathbb{Z})\ [x+y=0]$	For each integer x, there is an integer y such that $x+y=0$.
Negation	$(\exists x \in \mathbb{Z})\ (\forall y \in \mathbb{Z})\ [x+y\neq 0]$	There is an integer x such that for each integer y, we have $x+y\neq 0$.

Ex 2. What about interchange the order of two mixed quantifiers? circle one: yes or no

Ex 1. **Read** the Symbolically Write Guidelines, which are posted on the course Handout page.
Below are variants of statements from previous Exercises. For each Exercise:

O. Symbolically write (using quantifiers) the original statement.
Then indicate whether the original statement is true or false (no justification needed).

N. Symbolically write (using quantifiers) a useful negation of the original statement. Box your answer.
Then indicate whether the negation of the original statement is true or false (no justification needed).

1.1. If m is an odd integer, then $5m + 6$ is an even integer. ≈ER1.2.4b p27

1.2. If m and n are odd integers, then $mn + 7$ is an even integer. ER1.2.4c p27
I.e., The sum of 7 and the product of 2 odd integers is an even integer.

1.3. If a, b, and c are integers, then $ab^2 + a^3c^4 + a^5b^6c^7 + a^{100}b^{101}c^{102}$ is an odd integer. ≈ER1.2.7a p28