Theorem 1. Let $(x, y) \in \mathbb{R}^2$. If $x < -4$ and $y > 2$, then the distance between (x, y) and $(1, -2)$ is strictly larger than 6.

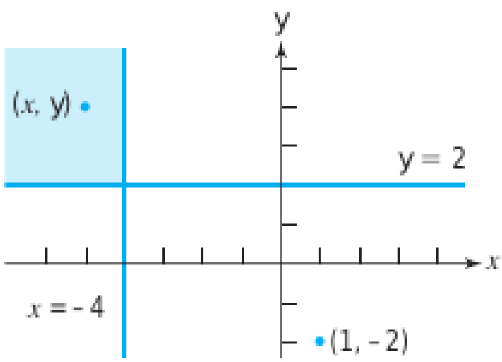
Instructions. Prove Thm. 1 algebraically ⟨using (in)equalities⟩. Do **not** use calculus. Do not argue geometrically but rather use geometric idea to form your Thinking Land.

Recall. The distance between $(x_1, y_1) \in \mathbb{R}^2$ and $(x_2, y_2) \in \mathbb{R}^2$, commonly denoted $d\left((x_1, y_1), (x_2, y_2)\right)$, is

$$d\left((x_1, y_1), (x_2, y_2)\right) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}\,.$$

Symbolically: $(\forall\, (x, y) \in \mathbb{R}^2)\ [\ (y < -4 \wedge y > 2) \implies d\left((x, y), (1, -2)\right) > 6\ ]$

Thinking Land. Let $(x, y) \in \mathbb{R}^2$. Given $x < -4$ and $y > 2$. WTS: $\sqrt{|x - 1|^2 + |y + 2|^2} > 6$.



- $x < -4 \;\Rightarrow\; x - 1 < -5 \;\Rightarrow\; |x - 1| > 5 \;\Rightarrow\; |x - 1|^2 > 25$
- $y > 2 \;\Rightarrow\; y + 2 > 4 \;\Rightarrow\; |y + 2| > 4 \;\Rightarrow\; |y + 2|^2 > 16$
- $\sqrt{|x - 1|^2 + |y + 2|^2} > \sqrt{25 + 16} = \sqrt{41} > \sqrt{36} = 6.$

Proof. Let $(x, y) \in \mathbb{R}^2$ with $x < -4$ and $y > 2$. Denote the distance between the two points $(x, y) \in \mathbb{R}^2$ and $(1, -2) \in \mathbb{R}^2$ by $d\left((x, y), (1, -2)\right)$ We shall show that

$$d\left((x, y), (1, -2)\right) > 6.$$

Recall that

$$d\left((x, y), (1, -2)\right) = \sqrt{|x - 1|^2 + |y + 2|^2}\,. \tag{1}$$

Since $x < -4$, we have that $x - 1 < -5$ and so ⟨$|x - 1| > 5$ so⟩

$$|x - 1|^2 > 25. \tag{2}$$

Since $y > 2$, we have that $y + 2 > 4$ and so ⟨$|y + 2| > 4$ so⟩

$$|y + 2|^2 > 16. \tag{3}$$

Thus equation (1), followed by inequalites (2) and (3) give

$$\begin{aligned} d\left((x, y), (1, -2)\right) &= \sqrt{|x - 1|^2 + |y + 2|^2} \\ &> \sqrt{25 + 16} \\ &= \sqrt{41} \\ &> \sqrt{36} \\ &= 6, \end{aligned}$$

which gives that $d\left((x, y), (1, -2)\right) > 6$, which is what we needed to show.

We have just shown that if $x < -4$ and $y > 2$, then $d\left((x, y), (1, -2)\right) > 6$. □