▷. Lemma POO is the book's Theorem 1.8 (p. 22).

Lemma POO. The product of two odd integers is an odd integer.

Symbolically: $(\forall x \in \mathbb{Z})\ (\forall y \in \mathbb{Z})\ [\ (x \text{ is odd } \wedge\ y \text{ is edd }) \implies xy \text{ is odd }]$

. . . .

Proof. Let x and y be odd integers. We will show that xy is an odd integer.

Since x is an odd integer, by definition of odd integer, there exists $k_x \in \mathbb{Z}$ such that

$$x = 2k_x + 1. \tag{1}$$

Since y is an odd integer, by definition of odd integer, there exists $k_y \in \mathbb{Z}$ such that

$$y = 2k_y + 1. \tag{2}$$

Using (1) and (2) and then using algebra we get

$$\begin{aligned} xy &= (2k_x + 1)\ (2k_y + 1) \\ &= 4k_xk_y + 2k_x + 2k_y + 1 \\ &= (2_x \cdot 2 \cdot k_xk_y + 2k_x + 2k_y) + 1 \\ &= 2\,(2k_xk_y + k_x + k_y) + 1 \end{aligned}$$

and letting $j = 2k_xk_y + k_x + k_y$

$$= 2j + 1.$$

Note $j \in \mathbb{Z}$ since $2, k_x, k_y \in \mathbb{Z}$ and the integers are closed under multiplication and addition. We have just show $xy = 2j + 1$ for some $j \in \mathbb{Z}$. Thus xy is odd by definition of odd. □

Theorem 1. Let $(x, y) \in \mathbb{R}^2$. If $x < -4$ and $y > 2$, then the distance between the points (x, y) and $(1, -2)$ is strictly larger than 6.

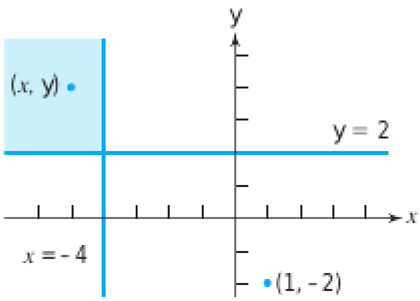
Instructions. Prove Thm. 1 algebraically ⟨using (in)equalities⟩. Do **not** use calculus. Do not argue geometrically but rather use geometric idea to form your Thinking Land.

Recall. The distance between $(x_1, y_1) \in \mathbb{R}^2$ and $(x_2, y_2) \in \mathbb{R}^2$, commonly denoted $d((x_1, y_1), (x_2, y_2))$, is

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}\,.$$

Symbolically: $(\forall (x, y) \in \mathbb{R}^2)\ [\ (y < -4 \wedge y > 2) \implies d((x, y), (1, -2)) > 6\ ]$

Thinking Land. Let $(x, y) \in \mathbb{R}^2$. Given $x < -4$ and $y > 2$. WTS: $\sqrt{|x-1|^2 + |y+2|^2} > 6$.



- $x < -4 \Rightarrow x - 1 < -5 \Rightarrow |x - 1| > 5 \Rightarrow |x - 1|^2 > 25$
- $y > 2 \Rightarrow y + 2 > 4 \Rightarrow |y + 2| > 4 \Rightarrow |y + 2|^2 > 16$
- $\sqrt{|x-1|^2 + |y+2|^2} > \sqrt{25 + 16} = \sqrt{41} > \sqrt{36} = 6.$

Proof. Let $(x, y) \in \mathbb{R}^2$ with $x < -4$ and $y > 2$. Denote the distance between the two points $(x, y) \in \mathbb{R}^2$ and $(1, -2) \in \mathbb{R}^2$ by $d((x, y), (1, -2))$. We shall show that

$$d((x, y), (1, -2)) > 6.$$

The distance formula gives

$$d((x, y), (1, -2)) = \sqrt{|x-1|^2 + |y+2|^2}\,. \tag{1}$$

Since $x < -4$, we have that $x - 1 < -5$ and so $|x - 1| > 5$. Thus

$$|x - 1|^2 > 25. \tag{2}$$

Since $y > 2$, we have that $y + 2 > 4$ and so $|y + 2| > 4$. Thus

$$|y + 2|^2 > 16. \tag{3}$$

Thus equation (1), followed by inequalites (2) and (3), give

$$\begin{aligned} d((x, y), (1, -2)) &= \sqrt{|x-1|^2 + |y+2|^2} \\ &> \sqrt{25 + 16} \\ &= \sqrt{41} \\ &> \sqrt{36} \\ &= 6. \end{aligned}$$

Thus $d((x, y), (1, -2)) > 6$.

We have just shown that if $(x, y) \in \mathbb{R}^2$ with $x < -4$ and $y > 2$, then $d((x, y), (1, -2)) > 6$. □