These *Practice Problems* are a sampling of the type of problems which could be an exam.
These problem are, in no way, meant as a comprehensive review for the exam.

Math is not a spectator sport.
Often we learn more from our failed attempts at a proof rather than reading a clean proof.
So give these problems a solid attempt before seeking help, e.g.:
looking through your notes and/or book, looking at the posted hints, or looking at Piazza.

You are highly encouraged to discuss and work together on these problems.
Our course Piazza page should help with this (let's use Piazza's `exam` folder).

- Post a note on Piazza saying, e.g., Fred has reserved TCL Study Room 123 for this Monday (7/4) for 7–9pm. Let me know if you want to join. Link to reserve a TCL Study Room on-line.
- Since the problems are not to be handed in (but rather serve as practice for exams), you may be sharing and posting not only ideas but also solutions. In Piazza (look at horizontal bar near top), the folder `piazza` contains a note about inserting a (PDF/JPEG) file.

▷. If divides is not in the material covered on the exam, then the following definition will be given on the exam.
Def. A nonzero integer n **divides** an integer b, denoted $n|b$, provided that $(\exists k \in \mathbb{Z})\ [nk = b]$.

▷. Recall the definition. A **counterexample** to a statement of the form $(\forall x \in U)\ [P(x)]$ is an object c in the universal set U for which $P(c)$ is false. So a counterexample is an example that shows $(\forall x \in U)\ [P(x)]$ is false.

The Practice Problems

★★★. **Instructions**. Below are several conjectures.

- If the conjecture is true, give a proof of the conjecture.
- If the conjecture is false, then give a counterexample to the conjecture. Explain why your counterexample does indeed show that the conjecture is false.

Conjecture A. For each natural number x strictly greater than 1, the quantity $x^2 + x + 41$ is prime.

Conjecture B. For each real number x, we have that $x + y = 0$ for some real number y.

Conjecture C. $(\forall x \in \mathbb{R})\ (\forall y \in \mathbb{R})\ [\ (x > 1 \wedge y > 0) \implies y^x > x]$

Conjecture D. For integers a, b, and c with $a \neq 0$, if a divides bc, then a divides b or a divides c.

Conjecture E. For integers a, b, c, and d with $a \neq 0$, if a divides $b - c$ and a divides $c - d$, then $a \mid (b - d)$.

Conjecture F. For each positive real number x, the inequality $x^2 - x \geq 0$ holds.

Conjecture G. For all positive real numbers x, we have $2^x > x + 1$.

Conjecture H. For every positive real number x, there is a positive real number y less than x with the property that for all positive real numbers z, we have $yz \geq z$.

Conjecture I. For every positive real number x, there is a positive real number y with the property that if $y < x$, then for all positive real numbers z, it holds that $yz \geq z$.

From: *A Transition To Advanced Mathematics* by Smith, Eggen, St. Andre. (6^{th}Ed.1.6.5/p54 = 7^{th}Ed.1.6.4/p57), which is over $300.

Hints

∘. When writing a proof, follow the proof Writing Guidelines while when writing symbolically, follow the Symbolically Write Guidelines. We have been using these guidelines throughout class.

∘. Avoid common mistakes.

- "x is a positive number" provided $x > 0$ while "x is a nonnegative number" provided $x \geq 0$
- "x is less than y" provided $x < y$ while "x is less than or equal to y" provided $x \leq y$
- don't forget needed parentheses, e.g., $a|b - 1$ does not make sense and should be written as $a|(b - 1)$.

Helpful Logical Equivalences.

When negating quantified statements, we often use the logical equivalencies (from our §2.2 Handout):

$$[\sim (P \Rightarrow Q)\ ] \ \equiv\ [\ P \wedge (\sim Q)\ ]$$ (how do you break a promise?)

$$[\sim (P \wedge Q)\ ] \ \equiv\ [\ P \Rightarrow (\sim Q)\ ]$$ (not in book)

$$[\sim (P \wedge Q)\ ] \ \equiv\ [\ (\sim P) \vee (\sim Q)\ ]$$ (De Morgans Law)

$$[\sim (P \vee Q)\ ] \ \equiv\ [\ (\sim P) \wedge (\sim Q)\ ]$$ (De Morgans Law) .

You should have a working knowledge of the logical equivalencies from this handout.

Samples of how the problems might look on an exam.

A. **Conjecture A**. For each natural number x strictly greater than 1, the quantity $x^2 + x + 41$ is prime.

A1. Complete the following two definitions.

- **Def**. A natural number p is a prime number provided ...
- **Def**. A natural number c is a composite number provided ...

A2. Symbolically write Conjecture A. As the universe, use $\mathbb{N}^{>1}$

A3. Symbolically write an useful negation (i.e., denial) of Conjecture A, without using $\sim$ nor $\neg$.

A4. Is Conjecture A true or false? (circle one) TRUE FALSE .

A5. Is the negation of Conjecture A true or false? (circle one) TRUE FALSE .

A6. Provide a proof or a counterexample to Conjecture A.

B. **Conjecture B**. For each real number x, we have that $x + y = 0$ for some real number y.

B1. Symbolically write Conjecture B.

B2. Symbolically write an useful negation of Conjecture B, without using $\sim$ nor $\neg$.

B3. Is Conjecture B true or false? (circle one) TRUE FALSE .

B4. Is the negation of Conjecture B true or false? (circle one) TRUE FALSE .

B5. Provide a proof or a counterexample to Conjecture B.

C. **Conjecture C**. $(\forall x \in \mathbb{R})\ (\forall y \in \mathbb{R})\ [\ (x > 1 \wedge y > 0) \implies y^x > x]$.

C1. Provide an equivalent symbolicaly written expression of Conjecture C by filling in the below line.

$$\left(\forall x \in \mathbb{R}^{>1}\right)\ \left(\forall b \in \mathbb{R}^{>0}\right)\ [\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\ ]\ .$$

C2. Provide an equivalent symbolicaly written expression of Conjecture C by filling in the below line.

$$\left(\forall b \in \mathbb{R}^{>0}\right)\ \left(\forall x \in \mathbb{R}^{>1}\right)\ [\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\ ]\ .$$

⚠. Recall, we can interchange two like quantifier (i.e., two $\forall$ together or two $\exists$ together) but we can NOT interchange two mixed quantifers (i.e., one $\forall$ and one $\exists$).

Hint. Think of b as the base of the expontential function.

C3. Is Conjecture C true or false? (circle one) TRUE FALSE .

C4. Is the negation of Conjecture C true or false? (circle one) TRUE FALSE .

C5. Provide a proof or a counterexample to Conjecture C.

D. **Conjecture D**. For integers a, b, and c with $a \neq 0$, if a divides bc, then a divides b or a divides c.

D1. If divides is not in the material covered on the exam, then the following definition will be given on the exam.

Def. A nonzero integer n **divides** an integer b, denoted $n|b$, provided that $(\exists k \in \mathbb{Z})\ [nk = b]$.

Otherwise, complete the below definition symbolically (not in English).

Def. A nonzero integer n **divides** an integer b, denoted $n|b$, provided that ___________________.

D2. Symbolically write Conjecture D. As the universe, use $\mathbb{R}^{\neq 0} \times \mathbb{R} \times \mathbb{R}$.

D3. Symbolically write an useful negation (i.e., denial) of Conjecture D, without using $\sim$ nor $\neg$.

D4. Is Conjecture D true or false? (circle one) TRUE FALSE .

D5. Is the negation of Conjecture D true or false? (circle one) TRUE FALSE .

D6. Provide a proof or a counterexample to Conjecture D.

E. **Conjecture E**. For integers a, b, c, and d with $a \neq 0$, if a divides $b - c$ and a divides $c - d$, then $a \mid (b - d)$.

E1. If divides is not in the material covered on the exam, then the following definition will be given on the exam.

Def. A nonzero integer n **divides** an integer b, denoted $n|b$, provided that $(\exists k \in \mathbb{Z})\ [nk = b]$.

Otherwise, complete the below definition symbolically (not in English).

Def. A nonzero integer n **divides** an integer b, denoted $n|b$, provided that ___________________.

E2. Symbolically write Conjecture E. As universe(s), use $\mathbb{Z}$, $\mathbb{Z}^{\neq 0}$ (also can be written $\mathbb{Z} \setminus \{0\}$), and/or some cross products of these.

E3. Symbolically write an useful negation of Conjecture E, without using $\sim$ nor $\neg$.

E4. Is Conjecture E true or false? (circle one) TRUE FALSE .

E5. Is the negation of Conjecture E true or false? (circle one) TRUE FALSE .

E6. Provide a proof or a counterexample to Conjecture E.

F. **Conjecture F**. For each positive real number x, the inequality $x^2 - x \geq 0$ holds.

F1. Symbolically write Conjecture F. As the universe use $\mathbb{R}^{>0}$.

F2. Symbolically write an useful negation of Conjecture F, without using $\sim$ nor $\neg$.

F3. Is Conjecture F true or false? (circle one) TRUE FALSE .

F4. Is the negation of Conjecture F true or false? (circle one) TRUE FALSE .

F5. Provide a proof or a counterexample to Conjecture F.

G. **Conjecture G**. For all positive real numbers x, we have $2^x > x + 1$.

G1. Symbolically write Conjecture G.

G2. Symbolically write an useful negation of Conjecture G, without using $\sim$ nor $\neg$.

G3. Is Conjecture G true or false? (circle one) TRUE FALSE .

G4. Is the negation of Conjecture G true or false? (circle one) TRUE FALSE .

G5. Provide a proof or a counterexample to Conjecture G.

H. **Conjecture H**. For every positive real number x, there is a positive real number y less than x with the property that for all positive real numbers z, we have $yz \geq z$.

H1. Symbolically write Conjecture H. As universes, use $\mathbb{R}^{>0}$

H2. Symbolically write an useful negation of Conjecture H, without using $\sim$ nor $\neg$ nor $\wedge$ nor $\vee$.

H3. Is Conjecture H true or false? (circle one) TRUE FALSE .

H4. Is the negation of Conjecture H true or false? (circle one) TRUE FALSE .

H5. Provide a proof or a counterexample to Conjecture H.

I. **Conjecture I**. For every positive real number x, there is a positive real number y with the property that if $y < x$, then for all positive real numbers z, it holds that $yz \geq z$.

I1. Symbolically write Conjecture I.

I2. Is Conjecture I true or false? (circle one) TRUE FALSE .

I3. Provide a proof or a counterexample to Conjecture I.