Strong Induction (also called complete induction, our book calls this 2nd PMI) §4.2 p194

Fix $n_0 \in \mathbb{Z}$.
If

BASE STEP: $P(n_0)$ is true

INDUCTIVE STEP: for each $n \in \mathbb{Z}^{\geq n_0}$: $\underbrace{[\ P(j) \text{ is true for } j \in \{n_0, 1+n_0, \dots, n\}\ ]}_{\text{inductive hypothesis}} \Rightarrow \underbrace{[\ P(n+1) \text{ is true}\ ]}_{\text{inductive conclusion}}$

then $P(n)$ is true for epz ach $n \in \mathbb{Z}^{\geq n_0}$.

Ex. **Theorem**. Let $\{a_n\}_{n=0}^{\infty}$ be the recursively defined sequence of integers

$$a_0 = 2 \tag{1}$$

$$a_1 = 4 \tag{2}$$

$$a_2 = 6 \tag{3}$$

and

$$a_n = 5a_{n-3} \quad \text{when} \quad n \in \mathbb{N} \text{ and } n \geq 3. \tag{4}$$

Then a_n is even for each $n \in \mathbb{Z}^{\geq 0} \overset{\text{i.e.}}{=} \{0, 1, 2, 3, 4, \dots\}$.

►. Symbolically: $(\forall n \in \mathbb{Z}^{\geq 0})\ \left[\ \left(a_0 = 2 \wedge a_1 = 4 \wedge a_2 = 6 \wedge \left(n \in \mathbb{N}^{\geq 3} \implies a_n = 5a_{n-3}\right)\right) \implies a_n \text{ is even}\ \right]$

Proof. Let $\{a_n\}_{n=0}^{\infty}$ be the recurively defined sequence of integers

$$a_0 = 2 \quad , \quad a_1 = 4 \quad , \quad a_2 = 6$$

and

$$a_n = 5a_{n-3} \quad \text{when} \quad n \in \mathbb{N} \text{ and } n \geq 3. \tag{RD}$$

We will show that a_n is even for each $n \in \mathbb{Z}^{\geq 0}$ by strong induction on n.

For the base step, first let $n = 0$. Then $a_n = a_0 = 2$, which is even. Next let $n = 1$ Then $a_n = a_1 = 4$, which is even. Finally let $n = 2$. Then $a_n = a_2 = 6$, which is even. Thus a_0, a_1, and a_2 are each even. This completes the base step.

For the inductive step, fix $n \in \mathbb{N}^{\geq 2}$ and assume the inductive hypothesis, which is

$$\text{if } j \in \{0, 1, 2, \dots, n\} \text{ then } a_j \text{ even.} \tag{IH}$$

We will show the inductive conclusion, which is

$$a_{n+1} \text{ is even.} \tag{IC}$$

Since $n \geq 2$,

$$n + 1 \geq 3$$

and so, by the recurive definition (RD) ⟨the recurive definition has *kicked in* for $n+1$ since $n+1 \geq 3$⟩

$$a_{n+1} = 5a_{(n+1)-3}$$

and so

$$a_{n+1} = 5a_{n-2} \tag{5}$$

Since $n \in \mathbb{Z}^{\geq 2}$ we have that $0 \leq n - 2 \leq n$ and so $n - 2 \in \{0, 1, 2, \dots, n\}$. Thus we can apply the inductive hypothesis (IH) to $j = n - 2$ to get that

$$a_{n-2} \text{ is even.} \tag{6}$$

Since the product of an even integer and any integer is an even integer [cf. Section 1.2 Exercise 3], equations (5) and (6) give that a_{n+1} is even. This completes the inductive step.

Thus the base step and the inductive step hold. So, by strong induction, the Theorem holds for all $n \in \mathbb{Z}^{\geq 0}$. □