<u>Strong Induction</u> (also called complete induction, our book calls this 2nd PMI) §4.2 p194

Fix $n_0 \in \mathbb{Z}$.
If

BASE STEP: $P(n_0)$ is true

INDUCTIVE STEP: for each $n \in \mathbb{Z}^{\geq n_0}$: $\underbrace{\left[\ P(j) \text{ is true for } j \in \{n_0, 1+n_0, \ldots, n\}\ \right]}_{\text{inductive hypothesis}} \Rightarrow \underbrace{\left[\ P(n+1) \text{ is true}\ \right]}_{\text{inductive conclusion}}$

then $P(n)$ is true for epz ach $n \in \mathbb{Z}^{\geq n_0}$.

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**Ex.** **Theorem**. Each natural number $n$ has a factorization as

$$n = 2^k m$$

for some $k$ is some nonnegative integer and some odd natural number $m$.

**►.** Written symbolically: $(\forall n \in \mathbb{N})\ (\exists k \in \mathbb{Z})\ (\exists m \in \mathbb{N})\ \left[k \geq 0\ \wedge\ m \text{ is odd}\ \ \wedge\ n = 2^k m\right]$.

*Proof.* We shall show that if $n \in \mathbb{N}$ then $n$ can be written as

$$n = 2^k m \qquad \text{for some } k \in \mathbb{N} \cup \{0\} \text{ and odd natural number } m \tag{1}$$

by strong induction on $n$.

For the base step, let $n = 1$. Then

$$n = 1 = 2^0 \cdot 1 = 2^k m$$

where $k = 0 \in \mathbb{N} \cup \{0\}$ and $m = 1 \in \mathbb{N}$ is odd. Thus (1) holds when $n = 1$. This completes the base step.

For the inductive step, fix $n \in \mathbb{N}$. Assume the inductive hypothesis, which is

$$\text{if } j \in \{1, 2, \ldots, n\}, \text{ the } j = 2^a b \text{ for some } a \in \mathbb{N} \cup \{0\} \text{ and odd natural number } b. \tag{IH}$$

We will show the inductive conclusion, which is

$$n + 1 = 2^k m \qquad \text{for some } k \in \mathbb{N} \cup \{0\} \text{ and odd natural number } m, \tag{IC}$$

by considering ⟨the only possible⟩ two cases: $n$ is even and $n$ is odd.

For the first case, let $n$ be an even integer. Then $n+1$ is odd and so

$$n + 1 = 2^0\,(n+1) = 2^k m$$

where $k = 0 \in \mathbb{N} \cup \{0\}$ and $m = n + 1$ is an odd integer. Thus (IC) holds for the first case. For the second case, let $n$ be an odd integer. Then $n+1$ is even; thus, there is $l \in \mathbb{N}$ such that

$$n + 1 = 2l. \tag{2}$$

Note that $l \in \{1, 2, \ldots, n\}$ since $l \in \mathbb{N}$ and

$$1 \leq l = \frac{n+1}{2} \leq \frac{n+n}{2} = n.$$

Thus by the inductive hypotheses (IH), applies to $l$, there exists $a \in \mathbb{N} \cup \{0\}$ and an odd natural number $b$ such that

$$l = 2^a b. \tag{3}$$

Equations (2) and (3) give,

$$n + 1 = 2l = 2\,(2^a b) = 2^{a+1} b = 2^k m$$

where $m{:=}b$ is an odd natural number and $k{:=}a+1 \in \mathbb{N} \cup \{0\}$ (since $a \in \mathbb{N} \cup \{0\}$). Thus (IC) holds for the second case. This completes the inductive step.

Thus the base step and the inductive step hold. So, by strong induction, equation (1) holds for all $n \in \mathbb{N}$. □