Strong Induction (also called complete induction, our book calls this 2nd PMI) §4.2 p194

Fix $n_0 \in \mathbb{Z}$.
If

BASE STEP: $P(n_0)$ is true

INDUCTIVE STEP: for each $n \in \mathbb{Z}^{\geq n_0}$: $\underbrace{[\ P(j) \text{ is true for } j \in \{n_0, 1+n_0, \ldots, n\}\ ]}_{\text{inductive hypothesis}} \Rightarrow \underbrace{[\ P(n+1) \text{ is true}\ ]}_{\text{inductive conclusion}}$

then $P(n)$ is true for each $n \in \mathbb{Z}^{\geq n_0}$.

Ex. **Theorem**. Let $\{a_n\}_{n=0}^{\infty}$ be the recursively defined sequence of integers

$$a_0 = 2 \quad , \quad a_1 = 4 \quad , \quad a_2 = 6$$

and

$$a_n = 5a_{n-3} \quad \text{when} \quad n \in \mathbb{N} \text{ and } n \geq 3.$$

Then a_n is even for each $n \in \mathbb{Z}^{\geq 0} \overset{\text{i.e.}}{=} \{0, 1, 2, 3, 4, \ldots\}$.

►. Symbolically: $(\forall n \in \mathbb{Z}^{\geq 0})\ \left[\ \left(a_0 = 2 \wedge a_1 = 4 \wedge a_2 = 6 \wedge \left(n \in \mathbb{N}^{\geq 3} \implies a_n = 5a_{n-3}\right)\right) \implies a_n \text{ is even}\ \right]$

Proof. Let $\{a_n\}_{n=0}^{\infty}$ be the recurively defined sequence of integers

$$a_0 = 2 \quad , \quad a_1 = 4 \quad , \quad a_2 = 6$$

and

$$a_n = 5a_{n-3} \quad \text{when} \quad n \in \mathbb{N} \text{ and } n \geq 3. \tag{RD}$$

We will show that a_n is even for each $n \in \mathbb{Z}^{\geq 0}$ by strong induction on n.

For the base step, first let $n = 0$. Then $a_n = a_0 = 2$, which is even. Next let $n = 1$ Then $a_n = a_1 = 4$, which is even. Finally let $n = 2$. Then $a_n = a_2 = 6$, which is even. Thus a_0, a_1, and a_2 are each even. This completes the base step.

For the inductive step, fix $n \in \mathbb{N}^{\geq 2}$ and assume the inductive hypothesis, which is

$$\text{if } j \in \{0, 1, 2, \ldots, n\} \text{ then } a_j \text{ even.} \tag{IH}$$

We will show the inductive conclusion, which is

$$a_{n+1} \text{ is even.} \tag{IC}$$

Since $n \geq 2$,

$$n + 1 \geq 3$$

and so, by the recurive definition (RD) ⟨the recurive definition has *kicked in* for a_{n+1} since $n + 1 \geq 3$⟩

$$a_{n+1} = 5a_{(n+1)-3}$$

and so

$$a_{n+1} = 5a_{n-2}. \tag{1}$$

Since $n \in \mathbb{Z}^{\geq 2}$, we know $2 \leq n$ and so

$$0 \leq n - 2 \leq n,$$

which gives $n - 2 \in \{0, 1, 2, \ldots, n\}$. Thus we can apply the inductive hypothesis (IH) to $j = n - 2$ to get

$$a_{n-2} \text{ is even.} \tag{2}$$

Since the product of an even integer and any integer is an even integer [cf. Section 1.2 Exercise 3], equations (1) and (2) give that a_{n+1} is even. This completes the inductive step.

Thus the base step and the inductive step hold. So, by strong induction, the Theorem holds for all $n \in \mathbb{Z}^{\geq 0}$. □

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Fix $n_0 \in \mathbb{Z}$.
If

BASE STEP: $P(n_0)$ is true

INDUCTIVE STEP: for each $n \in \mathbb{Z}^{\geq n_0}$: $\underbrace{[\ P(j) \text{ is true for } j \in \{n_0, 1+n_0, \ldots, n\}\ ]}_{\text{inductive hypothesis}} \Rightarrow \underbrace{[\ P(n+1) \text{ is true}\ ]}_{\text{inductive conclusion}}$

then $P(n)$ is true for each $n \in \mathbb{Z}^{\geq n_0}$.

Theorem. Each natural number n has a factorization as

$$n = 2^k m$$

for some k is some nonnegative integer and some odd natural number m.

►. Written symbolically: $(\forall n \in \mathbb{N})\ \left(\exists k \in \mathbb{Z}^{\geq 0}\right)\ (\exists m \in \mathbb{N})\ \left[m \text{ is odd } \ \wedge\ n = 2^k m\right]$.

Proof. We shall show that if $n \in \mathbb{N}$ then n can be written as

$$n = 2^k m \qquad \text{for some } k \in \mathbb{Z}^{\geq 0} \text{ and odd natural number } m. \tag{3}$$

by strong induction on n.

For the base step, let $n = 1$. Then

$$n = 1 = 2^0 \cdot 1 = 2^k m$$

where $k = 0 \in \mathbb{Z}^{\geq 0}$ and $m = 1 \in \mathbb{N}$ is odd. So (3) holds when $n = 1$. This completes the base step.

For the inductive step, fix $n \in \mathbb{N}$. Assume the inductive hypothesis, which is

$$\text{if } j \in \{1, 2, \ldots, n\} \text{ then} \tag{IH}$$
$$j = 2^{k_j} m_j \quad \text{for some } k_j \in \mathbb{Z}^{\geq 0} \text{ and odd natural number } m_j.$$

We will show the inductive conclusion, which is

$$n + 1 = 2^k m \qquad \text{for some } k \in \mathbb{Z}^{\geq 0} \text{ and odd natural number } m, \tag{IC}$$

by considering ⟨the only possible⟩ two cases: n is even and n is odd.

For the first case, let n be an even natural number. Then $n+1$ is an odd natural numbers so

$$n + 1 = 2^0\,(n+1) = 2^k m$$

where $k = 0 \in \mathbb{Z}^{\geq 0}$ and $m = n + 1$ is an odd natural number. Thus (IC) holds for the first case.

For the second case, let n be an odd natural number. Then $n+1$ is an even natural number; thus, there is $l \in \mathbb{N}$ such that

$$n + 1 = 2l. \tag{4}$$

Note that $l \in \{1, 2, \ldots, n\}$ since $l \in \mathbb{N}$ and

$$1 \leq l = \frac{n+1}{2} \leq \frac{n+n}{2} = n.$$

Thus by the inductive hypotheses (IH) ⟨think $j := l$⟩ there exists $k_l \in \mathbb{Z}^{\geq 0}$ and an odd natural number m_l such that

$$l = 2^{k_l} m_l. \tag{5}$$

Equations (4) and (5) give,

$$n + 1 = 2l = 2\left(2^{k_l} m_l\right) = \left(2^{1+k_l}\right)\,(m_l) = 2^k m$$

where $m{:=}m_l$ is an odd natural number and $k{:=}1 + k_l \in \mathbb{Z}^{\geq 0}$ (since $k_l \in \mathbb{Z}^{\geq 0}$). Thus (IC) holds for the second case. This completes the inductive step.

Thus the base step and the inductive step hold. So, by strong induction, equation (1) holds for all $n \in \mathbb{N}$. □