When writing an induction proof, remember to *keep your reader informed*; thus, you should:

(1) say what you are trying to show inductively
(2) say what your induction variable is (e.g., if you are trying to show $(\forall j \in \mathbb{N})\,[P(j)]$ then say: We will show that *blub* holds for each $j \in \mathbb{N}$ by induction on j.)
(3) indicate where your base step begins
(4) indicate where your base step ends
(5) indicate where your inductive step begins
(6) clearly state your inductive hypothesis (IH)
(7) clearly state your inductive conclusion (IC)
(8) indicate where your inductive step ends.

As with any proof, clean up your *Thinking Land* so that you

(∗) do NOT pull your reader therough the mud with you.

Ex. Show that for $n \in \mathbb{N}$ with $n \geq 6$

$$n^3 < n!\,.$$

First Proof. We shall show that for each $n \in \mathbb{N}^{\geq 6}$

$$n^3 < n! \tag{1}$$

by induction on n.

For the base step, let $n = 6$. Then

$$n^3 = 6^3 = 216. \tag{2}$$

while

$$n! = 6! = 720. \tag{3}$$

Since $216 < 720$, the inequality in (1) holds when $n = 6$. This completes the base step.

For the inductive step, fix a natural number $n \in \mathbb{N}^{\geq 6}$. Assume that

$$n^3 < n!. \tag{IH}$$

We need to show that

$$(n+1)^3 < (n+1)!. \tag{IC}$$

Thus

$$(n+1)! \;=\; (n+1)\ [\ n!\ ]$$

and by the inductive hypotheses (IH)

$$> (n+1)\ \left[\ n^3\ \right]$$
$$= (n+1)\ n \cdot n^2.$$

and since $n \geq 6 \geq 4$

$$\geq (n+1)\ 4 \cdot n^2.$$
$$= (n+1)\ (2n)^2$$
$$= (n+1)\ (n+n)^2$$

and since $n \in \mathbb{N}$ so $n \geq 1$

$$\geq\ (n+1)\ (n+1)^2$$
$$= (n+1)^3\,.$$

Thus inequality (IC) hold. This completes the inductive step.

Thus, by induction, inequality (1) holds for each natural number $n \in \mathbb{N}^{\geq 6}$. □☺☺.

$*$. Show that for $n \in \mathbb{N}$ with $n \geq 6$

$$n^3 < n!\ .$$

Second Proof

Proof We shall show that for each $n \in \mathbb{N}^{\geq 6}$

$$n^3 < n! \tag{1}$$

by induction on n.

For the base step, let $n = 6$. Then

$$n^3 = 6^3 = 216 < 720 = 6! = n!.$$

Thus inequality (1) holds when $n = 6$.

For the inductive step, fix a natural number $n \in \mathbb{N}^{\geq 6}$. Assume that

$$n^3 < n!. \tag{IH}$$

We need to show that

$$(n+1)^3 < (n+1)!. \tag{IC}$$

⟨Note (IC) is equivalent to $(n+1)^3 < (n+1)\,[\underset{\sim}{n!}].$ ⟩

Using the inductive hypothesis (IH) and algebra, we have

$$(n+1)^3 = (n+1)\left[(n+1)^2\right]$$

⟨would be done if $\left[(n+1)^2\right] < n!$ ... looking at our (IH) maybe we can go for $\left[(n+1)^2\right] \overset{\text{go for}}{\leq} n^3$ for then we would have $\left[(n+1)^2\right] \overset{\text{go for}}{\leq} n^3 \overset{\text{(IH)}}{<} n!$ ... and $\left[(n+1)^2\right] \overset{\text{go for}}{\leq} n^3$ seems plausible for n big enough⟩

and since $1 < 6 \leq n$,

$$\begin{aligned} &< (n+1)\left[(n+n)^2\right] \\ &= (n+1)\left[4n^2\right] \end{aligned}$$

and since $4 < 6 \leq n$,

$$\begin{aligned} &< (n+1)\left(n \cdot n^2\right) \\ &= (n+1)\left(n^3\right) \end{aligned}$$

and using our inductive hypothesis (IH),

$$\begin{aligned} &< (n+1)\,(n!) \\ &= (n+1)!. \end{aligned}$$

Thus inequality (IC) hold. This completes the inductive step.

Thus, by induction, inequality (1) holds for each natural number $n \in \mathbb{N}^{\geq 6}$. □☺☺.