Induction (basic form). Let $P(n)$ be an open sentence in the variable $n \in \mathbb{N}$. §4.1 p173

If

BASE STEP: $P(1)$ is true

INDUCTIVE STEP: for each $n \in \mathbb{N}$: $\underbrace{[\ P(n) \text{ is true}\ ]}_{\text{inductive hypothesis}} \implies \underbrace{[\ P(n+1) \text{ is true}\ ]}_{\text{inductive conclusion}}$

then $P(n)$ is true for each $n \in \mathbb{N}$.

Ex. Show that $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$ for each integer n.

Proof. For each $n \in \mathbb{N}$, let $P(n)$ be the statement

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}. \tag{1}$$

We will show that $P(n)$ is true for each $n \in \mathbb{N}$ by induction on n.

For the base step, we shall show that $P(n)$ holds when $n = 1$. Let $n = 1$. Then

$$\sum_{i=1}^{n} \frac{1}{i^2} = \sum_{i=1}^{1} \frac{1}{i^2} = \frac{1}{1^2} = 1 = 2 - 1 \leq 2 - \frac{1}{1} = 2 - \frac{1}{n}.$$

Thus $P(1)$ is true. This finishes the base step.

For the inductive step, fix $n \in \mathbb{N}$. We assume the inductive hypothesis, which is $\langle P(n)$ is true$\rangle$

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}. \tag{IH}$$

For the inductive step, your goal is to show the inductive conclusion, which is $\langle$i.e., $P(n+1)$ is true$\rangle$

$$\sum_{i=1}^{n+1} \frac{1}{i^2} \leq 2 - \frac{1}{n+1}. \tag{IC}$$

Using the inductive hypothesis $\langle$recall $\sum_{i=1}^{n+1} a_i = [a_1 + a_2 + \cdots + a_n] + a_{n+1} = \left[\sum_{i=1}^{n} a_i\right] + a_{n+1}\rangle$ and then algebra we get

$$\sum_{i=1}^{n+1} \frac{1}{i^2} = \left[\sum_{i=1}^{n} \frac{1}{i^2}\right] + \frac{1}{(n+1)^2}$$

and by the inductive hypothesis (IH)

$$\leq \left[2 - \frac{1}{n}\right] + \frac{1}{(n+1)^2}$$

$\langle$look at (IC) for hint on where to go next$\rangle$

$$\begin{aligned} &= 2 - \left[\frac{1}{n} - \frac{1}{(n+1)^2}\right] \\ &= 2 - \left[\frac{(n+1)^2 - n}{n(n+1)^2}\right] \\ &= 2 - \left[\frac{n^2 + n + 1}{n(n+1)^2}\right] \end{aligned}$$

$\langle$look again at (IC) for another hint on where to go next$\rangle$

$$\begin{aligned} &= 2 - \left(\frac{1}{n+1}\right)\left(\frac{n^2+n+1}{n(n+1)}\right) \\ &= 2 - \underbrace{\left(\frac{1}{n+1}\right)}_{x}\underbrace{\left(\frac{n^2+n+1}{n^2+n}\right)}_{h} \quad \langle \overset{\text{for (IC)}}{\underset{\text{want}}{\leq}} 2 - x\rangle \end{aligned}$$

$\langle$look at (IC) again ... have $\sum \leq 2 - xh$ but want $\sum \leq 2 - x$. Well: $2 - xh \leq 2 - x \Leftrightarrow -xh \leq -x \overset{\text{know}}{\underset{x>0}{\Leftrightarrow}} -h \leq -1 \Leftrightarrow h \geq 1\rangle$

$$= 2 - \left(\frac{1}{n+1}\right)\left(1 + \frac{1}{n^2+n}\right)$$

and since $n \geq 1$ we know that $\frac{1}{n+1} > 0$ and $1 + \frac{1}{n^2+n} \geq 1$ and so

$$\leq 2 - \left(\frac{1}{n+1}\right).$$

Thus (IC) hold. This completes the inductive step.

Thus, by induction, (1) holds for each $n \in \mathbb{N}$. □