Ex1. Prove that $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$ for each integer n.

WTS. $(\forall n \in \mathbb{N})\,[P(n) \text{ is true}]$ where $P(n)$ is the open sentence $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$ in the variable $n \in \mathbb{N}$.

Proof. Using basic induction on the variable n, we will show that for each $n \in \mathbb{N}$

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}. \tag{1}$$

For the base step, let $n = 1$. Since, when $n = 1$,

$$\sum_{i=1}^{n} \frac{1}{i^2} = \sum_{i=1}^{1} \frac{1}{i^2} = \frac{1}{1^2} = 1 \qquad \text{and} \qquad 2 - \frac{1}{n} = 2 - \frac{1}{1} = 2 - 1 = 1,$$

inequality (1) holds when $n = 1$. This finishes the base step.

For the inductive step, fix $n \in \mathbb{N}$. We assume the inductive hypothesis, which is $\langle P(n) \text{ is true} \rangle$

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}. \tag{IH}$$

For the inductive step, your goal is to show the inductive conclusion, which is $\langle P(n+1) \text{ is true} \rangle$

$$\sum_{i=1}^{n+1} \frac{1}{i^2} \leq 2 - \frac{1}{n+1}. \tag{IC}$$

We now have $\langle$ recall $\sum_{i=1}^{n+1} a_i = (a_1 + a_2 + \cdots + a_n) + a_{n+1} = \left(\sum_{i=1}^{n} a_i\right) + a_{n+1}$ $\rangle$

$$\sum_{i=1}^{n+1} \frac{1}{i^2} = \left(\sum_{i=1}^{n} \frac{1}{i^2}\right) + \frac{1}{(n+1)^2}$$

and by the inductive hypothesis (IH)

$$\leq \left(2 - \frac{1}{n}\right) + \frac{1}{(n+1)^2}$$

$\langle$whenever do not know what to do next, LOOK at (IC) for hint on where to go next$\rangle$

$$= 2 - \left[\frac{1}{n} - \frac{1}{(n+1)^2}\right]$$

$$= 2 - \left[\frac{(n+1)^2 - n}{n(n+1)^2}\right]$$

$$= 2 - \left(\frac{1}{n+1}\right)\left[\frac{n^2+n+1}{n(n+1)}\right]$$

$$= 2 - \left(\frac{1}{n+1}\right)\left[\frac{n^2+n+1}{n^2+n}\right]$$

(inequality help: $n^2+n+1 \boxed{\geq} n^2+n$ so $\frac{n^2+n+1}{n^2+n} \boxed{\geq} \frac{n^2+n}{n^2+n}$ so $-\left(\frac{1}{n+1}\right)\frac{n^2+n+1}{n^2+n} \boxed{\leq} -\left(\frac{1}{n+1}\right)\frac{n^2+n}{n^2+n}$)

$$\leq 2 - \left(\frac{1}{n+1}\right)\left[\frac{n^2+n}{n^2+n}\right]$$

$$= 2 - \left(\frac{1}{n+1}\right).$$

Thus (IC) hold. This completes the inductive step.

Thus, by induction, (1) holds for each $n \in \mathbb{N}$. $\square$

Rmk. When we write an induction proof, we usally write the Base Step first.
However, in your *Thinking Land*, we usually do the Inductive Step first. Why?
Let's say we want to show a $\left(\forall n \in \mathbb{Z}^{\geq 5}\right)[\,P(n)\,]$ and our inductive step (i.e., $P(n) \implies P(n+1)$) only works when $n \geq 7$ (and our inductive step just does not work when n is 5 or 6). All is not lost! In this situation, we need to show the base step $P(n)$ hold true when n is: 5, 6, and 7 .

Ex2. Prove that for $n \in \mathbb{N}$ with $n \geq 6$

$$n^3 < n!\,.$$

Proof. We shall show that for each $n \in \mathbb{N}^{\geq 6}$

$$n^3 < n! \tag{1}$$

by ⟨extended/generalized⟩ induction on n.

For the base step, let $n = 6$. Then

$$n^3 = 6^3 = 216. \tag{2}$$

while

$$n! = 6! = 720. \tag{3}$$

Since $216 < 720$, the inequality in (1) holds when $n = 6$. This completes the base step.

For the inductive step, fix a natural number $n \in \mathbb{N}^{\geq 6}$. Assume that

$$n^3 < n!. \tag{IH}$$

We need to show that

$$(n+1)^3 < (n+1)!. \tag{IC}$$

We now compute:

$$(n+1)! \;=\; (n+1)\;[\;n!\;]$$

and by the inductive hypotheses (IH)

$$> (n+1)\;\left[\;n^3\;\right]$$

⟨Look at (IC), which holds if $n^3 \overset{\text{go for}}{\geq} (n+1)^2$. Since $6 \leq n$, we know $(n+1)^2 \leq (n+n)^2 = (2n)^2 = 4n^2 \leq 6n^2 \leq n \cdot n^2 = n^3$.⟩

$$= (n+1)\; n \cdot n^2$$

and since $n \geq 6 \geq 4$

$$\geq (n+1)\; 4 \cdot n^2$$

$$= (n+1)\; (2n)^2$$

$$= (n+1)\; (n+n)^2$$

and since $n \in \mathbb{N}$ so $n \geq 1$

$$\geq\; (n+1)\; (n+1)^2$$

$$= (n+1)^3\,.$$

Thus inequality (IC) hold. This completes the inductive step.

Thus, by induction, inequality (1) holds for each natural number $n \in \mathbb{N}^{\geq 6}$. □☺☺.

Strong Induction (also called complete induction, our book calls this 2^{nd} PMI) §4.2 p194

Fix $n_0 \in \mathbb{Z}$.
If

BASE STEP: $P(n_0)$ is true

INDUCTIVE STEP: for each $n \in \mathbb{Z}^{\geq n_0}$: $\underbrace{[\ P(j) \text{ is true for } j \in \{n_0, 1+n_0, \ldots, n\}\ ]}_{\text{inductive hypothesis}} \Rightarrow \underbrace{[\ P(n+1) \text{ is true}\ ]}_{\text{inductive conclusion}}$

then $P(n)$ is true for each $n \in \mathbb{Z}^{\geq n_0}$.

Ex3. Let $\{a_n\}_{n=0}^{\infty}$ be the recursively defined sequence of integers

$$a_0 = 2 \quad , \quad a_1 = 4 \quad , \quad a_2 = 6$$

and

$$a_n = 5a_{n-3} \quad \text{when} \quad n \in \mathbb{N} \text{ and } n \geq 3. \qquad \text{(RD)}$$

Prove that a_n is even for each $n \in \mathbb{Z}^{\geq 0} \overset{\text{i.e.}}{=} \{0, 1, 2, 3, 4, \ldots\}$. RD = Recursive Def. ↑

▸. Symbolically:

$$(\forall n \in \mathbb{Z}^{\geq 0}) \left[\left(a_0 = 2 \wedge a_1 = 4 \wedge a_2 = 6 \wedge \left(n \in \mathbb{N}^{\geq 3} \implies a_n = 5a_{n-3}\right)\right) \implies a_n \text{ is even} \right]$$

Thinking Land

Let's make a chart to help us understand better what is going on.

n	a_n
0	$a_0 = 2$ (given)
1	$a_1 = 4$ (given)
2	$a_2 = 6$ (given)
	now the recursive definition *kicks in* that $a_n = 5a_{n-3}$
3	$a_3 = 5a_{3-3} = 5a_0 = 5(2) = 10$
4	$a_4 = 5a_{4-3} = 5a_1 = 5(4) = 20$
5	$a_5 = 5a_{5-3} = 5a_2 = 5(6) = 30$
6	$a_6 = 5a_{6-3} = 5a_3 = 5(10) = 50$
7	$a_7 = 5a_{7-3} = 5a_4 = 5(20) = 100$
8	$a_8 = 5a_{8-3} = 5a_5 = 5(30) = 150$
	Do we see a pattern?

◦. For the Base Step, which n's do we need to check? $n = 0, 1, 2$.

◦. Since in the Base Step we verified the Thm. holds up to (and including) $n =$ 2,

where should we start the Induction Step? At $n =$ 2.

So the first line in your induction step should look something line:

For the inductive step, fix $n \in \mathbb{N}$ such that $n \geq$ 2. Assume the inductive hypothesis, which is

$$\text{if } j \in \{0, 1, 2, \ldots, n\} \text{ then } a_j \text{ even.} \qquad \text{(IH)}$$

We will show the inductive conclusion, which is

$$a_{n+1} \text{ is even.} \qquad \text{(IC)}$$

Strong Induction. §4.2 p194

Fix $n_0 \in \mathbb{Z}$.

If

BASE STEP: $P(n_0)$ is true

INDUCTIVE STEP: for each $n \in \mathbb{Z}^{\geq n_0}$: $\underbrace{[\ P(j) \text{ is true for } j \in \{n_0, 1+n_0, \ldots, n\}\ ]}_{\text{inductive hypothesis}} \Rightarrow \underbrace{[\ P(n+1) \text{ is true}\ ]}_{\text{inductive conclusion}}$

then $P(n)$ is true for each $n \in \mathbb{Z}^{\geq n_0}$.

Ex3. Let $\{a_n\}_{n=0}^{\infty}$ be the recursively defined sequence of integers

$$a_0 = 2 \quad , \quad a_1 = 4 \quad , \quad a_2 = 6$$

and

$$a_n = 5a_{n-3} \quad \text{when} \quad n \in \mathbb{N} \text{ and } n \geq 3. \tag{RD}$$

Prove that a_n is even for each $n \in \mathbb{Z}^{\geq 0} \overset{\text{i.e.}}{=} \{0, 1, 2, 3, 4, \ldots\}$.

►. Symbolically: $(\forall n \in \mathbb{Z}^{\geq 0})\ \left[\ \left(a_0 = 2 \wedge a_1 = 4 \wedge a_2 = 6 \wedge \left(n \in \mathbb{N}^{\geq 3} \implies a_n = 5a_{n-3}\right)\right) \implies a_n \text{ is even}\ \right]$

Proof. Let $\{a_n\}_{n=0}^{\infty}$ be the recursively defined sequence of integers

$$a_0 = 2 \quad , \quad a_1 = 4 \quad , \quad a_2 = 6$$

and

$$a_n = 5a_{n-3} \quad \text{when} \quad n \in \mathbb{N} \text{ and } n \geq 3. \tag{RD}$$

We will show that a_n is even for each $n \in \mathbb{Z}^{\geq 0}$ by strong induction on n.

For the base step, first let $n = 0$. Then $a_n = a_0 = 2$, which is even. Next let $n = 1$ Then $a_n = a_1 = 4$, which is even. Finally let $n = 2$. Then $a_n = a_2 = 6$, which is even. Thus a_0, a_1, and a_2 are each even. This completes the base step.

For the inductive step, fix $n \in \mathbb{N}^{\geq 2}$ and assume the inductive hypothesis, which is

$$\text{if } j \in \{0, 1, 2, \ldots, n\} \text{ then } a_j \text{ even.} \tag{IH}$$

We will show the inductive conclusion, which is

$$a_{n+1} \text{ is even.} \tag{IC}$$

Since $n \geq 2$,

$$n + 1 \geq 3$$

and so, by the recursive definition (RD) ⟨the recurive definition has *kicked in* for a_{n+1} since $n + 1 \geq 3$⟩

$$a_{n+1} = 5a_{(n+1)-3}$$

and so

$$a_{n+1} = 5a_{n-2}. \tag{4}$$

Since $n \in \mathbb{Z}^{\geq 2}$, we know $2 \leq n$ and so

$$0 \leq n - 2 \leq n,$$

which gives $n - 2 \in \{0, 1, 2, \ldots, n\}$. Thus we can apply the inductive hypothesis (IH) to $j = n - 2$ to get

$$a_{n-2} \text{ is even.} \tag{5}$$

Since the product of an even integer and any integer is an even integer (by Lemma PEA), equations (4) and (5) give that a_{n+1} is even. This completes the inductive step.

Thus the base step and the inductive step hold. So, by strong induction, the Example holds for all $n \in \mathbb{Z}^{\geq 0}$. □