

Ex1. Prove that $\sum_{i=1}^n \frac{1}{i^2} \leq 2 - \frac{1}{n}$ for each integer n .

wts. $(\forall n \in \mathbb{N}) [P(n) \text{ is true}]$ where $P(n)$ is the open sentence $\sum_{i=1}^n \frac{1}{i^2} \leq 2 - \frac{1}{n}$ in the variable $n \in \mathbb{N}$.

Proof. Using basic induction on the variable n , we will show that for each $n \in \mathbb{N}$

$$\sum_{i=1}^n \frac{1}{i^2} \leq 2 - \frac{1}{n}. \quad (1)$$

For the base step, let $n = 1$. Since, when $n = 1$,

$$\sum_{i=1}^n \frac{1}{i^2} = \sum_{i=1}^1 \frac{1}{i^2} = \frac{1}{1^2} = 1 \quad \text{and} \quad 2 - \frac{1}{n} = 2 - \frac{1}{1} = 2 - 1 = 1,$$

inequality (1) holds when $n = 1$. This finishes the base step.

For the inductive step, fix $n \in \mathbb{N}$. We assume the inductive hypothesis, which is $\langle P(n) \text{ is true} \rangle$

$$\sum_{i=1}^n \frac{1}{i^2} \leq 2 - \frac{1}{n}. \quad (\text{IH})$$

For the inductive step, your goal is to show the inductive conclusion, which is $\langle P(n+1) \text{ is true} \rangle$

$$\sum_{i=1}^{n+1} \frac{1}{i^2} \leq 2 - \frac{1}{n+1}. \quad (\text{IC})$$

We now have $\langle \text{recall } \sum_{i=1}^{n+1} a_i = (a_1 + a_2 + \cdots + a_n) + a_{n+1} = \left(\sum_{i=1}^n a_i \right) + a_{n+1} \rangle$

$$\sum_{i=1}^{n+1} \frac{1}{i^2} = \left(\sum_{i=1}^n \frac{1}{i^2} \right) + \frac{1}{(n+1)^2}$$

and by the inductive hypothesis (IH)

$$\leq \left(2 - \frac{1}{n} \right) + \frac{1}{(n+1)^2}$$

$\langle \text{whenever do not know what to do next, LOOK at (IC) for hint on where to go next} \rangle$

$$\begin{aligned} &= 2 - \left[\frac{1}{n} - \frac{1}{(n+1)^2} \right] \\ &= 2 - \left[\frac{(n+1)^2 - n}{n(n+1)^2} \right] \\ &= 2 - \left(\frac{1}{n+1} \right) \left[\frac{n^2 + n + 1}{n(n+1)} \right] \\ &= 2 - \left(\frac{1}{n+1} \right) \left[\frac{n^2 + n + 1}{n^2 + n} \right] \end{aligned}$$

(inequality help: $n^2 + n + 1 \geq n^2 + n$ so $\frac{n^2+n+1}{n^2+n} \geq \frac{n^2+n}{n^2+n}$ so $-\left(\frac{1}{n+1}\right) \frac{n^2+n+1}{n^2+n} \leq -\left(\frac{1}{n+1}\right) \frac{n^2+n}{n^2+n}$)

$$\begin{aligned} &\leq 2 - \left(\frac{1}{n+1} \right) \left[\frac{n^2 + n}{n^2 + n} \right] \\ &= 2 - \left(\frac{1}{n+1} \right). \end{aligned}$$

Thus (IC) hold. This completes the inductive step.

Thus, by induction, (1) holds for each $n \in \mathbb{N}$. □

Rmk. When we write an induction proof, we usually write the Base Step first.

However, in your *Thinking Land*, we usually do the Inductive Step first. Why?

Let's say we want to show a $(\forall n \in \mathbb{Z}^{\geq 5}) [P(n)]$ and our inductive step (i.e., $P(n) \implies P(n+1)$) only works when $n \geq 7$ (and our inductive step just does not work when n is 5 or 6). All is not lost! In this situation, we need to show the base step $P(n)$ hold true when n is: 5, 6, and 7.

Ex2. Prove that for $n \in \mathbb{N}$ with $n \geq 6$

$$n^3 < n!.$$

Proof. We shall show that for each $n \in \mathbb{N}^{\geq 6}$

$$n^3 < n! \tag{1}$$

by $\langle$ extended/generalized $\rangle$ induction on n .

For the base step, let $n = 6$. Then

$$n^3 = 6^3 = 216. \tag{2}$$

while

$$n! = 6! = 720. \tag{3}$$

Since $216 < 720$, the inequality in (1) holds when $n = 6$. This completes the base step.

For the inductive step, fix a natural number $n \in \mathbb{N}^{\geq 6}$. Assume that

$$n^3 < n!. \tag{IH}$$

We need to show that

$$(n+1)^3 < (n+1)!. \tag{IC}$$

We now compute:

$$(n+1)! = (n+1) [n!]$$

and by the inductive hypotheses (IH)

$$> (n+1) [n^3]$$

$\langle$ Look at (IC), which holds if $n^3 \overset{\text{go for}}{\geq} (n+1)^2$. Since $6 \leq n$, we know $(n+1)^2 \leq (n+n)^2 = (2n)^2 = 4n^2 \leq 6n^2 \leq n \cdot n^2 = n^3$. $\rangle$

$$= (n+1) n \cdot n^2$$

and since $n \geq 6 \geq 4$

$$\begin{aligned} &\geq (n+1) 4 \cdot n^2 \\ &= (n+1) (2n)^2 \\ &= (n+1) (n+n)^2 \end{aligned}$$

and since $n \in \mathbb{N}$ so $n \geq 1$

$$\begin{aligned} &\geq (n+1) (n+1)^2 \\ &= (n+1)^3. \end{aligned}$$

Thus inequality (IC) hold. This completes the inductive step.

Thus, by induction, inequality (1) holds for each natural number $n \in \mathbb{N}^{\geq 6}$.

□😊😊.

Strong Induction (also called complete induction, our book calls this 2nd PMI)

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Fix $n_0 \in \mathbb{Z}$.

If

BASE STEP: $P(n_0)$ is true

INDUCTIVE STEP: for each $n \in \mathbb{Z}^{\geq n_0}$: $\underbrace{[P(j) \text{ is true for } j \in \{n_0, 1 + n_0, \dots, n\}]}_{\text{inductive hypothesis}} \Rightarrow \underbrace{[P(n+1) \text{ is true}]}_{\text{inductive conclusion}}$

then $P(n)$ is true for each $n \in \mathbb{Z}^{\geq n_0}$.

Ex3. Let $\{a_n\}_{n=0}^{\infty}$ be the recursively defined sequence of integers

$$a_0 = 2 \quad , \quad a_1 = 4 \quad , \quad a_2 = 6$$

and

$$a_n = 5a_{n-3} \quad \text{when } n \in \mathbb{N} \text{ and } n \geq 3. \quad (\text{RD})$$

Prove that a_n is even for each $n \in \mathbb{Z}^{\geq 0} \stackrel{\text{i.e.}}{=} \{0, 1, 2, 3, 4, \dots\}$.

RD = Recursive Def. $\uparrow$

►. Symbolically:

$$(\forall n \in \mathbb{Z}^{\geq 0}) \left[(a_0 = 2 \wedge a_1 = 4 \wedge a_2 = 6 \wedge (n \in \mathbb{N}^{\geq 3} \Rightarrow a_n = 5a_{n-3})) \Rightarrow a_n \text{ is even} \right]$$

Thinking Land

Let's make a chart to help us understand better what is going on.

n	a_n
0	$a_0 = 2$ (given)
1	$a_1 = 4$ (given)
2	$a_2 = 6$ (given)
now the recursive definition <i>kicks in</i> that $a_n = 5a_{n-3}$	
3	$a_3 = 5a_{3-3} = 5a_0 = 5(2) = 10$
4	$a_4 = 5a_{4-3} = 5a_1 = 5(4) = 20$
5	$a_5 = 5a_{5-3} = 5a_2 = 5(6) = 30$
6	$a_6 = 5a_{6-3} = 5a_3 = 5(10) = 50$
7	$a_7 = 5a_{7-3} = 5a_4 = 5(20) = 100$
8	$a_8 = 5a_{8-3} = 5a_5 = 5(30) = 150$
Do we see a pattern?	

◦. For the Base Step, which n 's do we need to check? $n = 0, 1, 2$.

◦. Since in the Base Step we verified the Thm. holds up to (and including) $n = \underline{2}$, where should we start the Induction Step? At $n = \underline{2}$.

So the first line in your induction step should look something like:

For the inductive step, fix $n \in \mathbb{N}$ such that $n \geq \underline{2}$. Assume the inductive hypothesis, which is

$$\text{if } j \in \{0, 1, 2, \dots, n\} \text{ then } a_j \text{ even.} \quad (\text{IH})$$

We will show the inductive conclusion, which is

$$a_{n+1} \text{ is even.} \quad (\text{IC})$$

Strong Induction.Fix $n_0 \in \mathbb{Z}$.§4.2
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If

BASE STEP: $P(n_0)$ is trueINDUCTIVE STEP: for each $n \in \mathbb{Z}^{\geq n_0}$: $\underbrace{[P(j) \text{ is true for } j \in \{n_0, 1 + n_0, \dots, n\}]}_{\text{inductive hypothesis}} \Rightarrow \underbrace{[P(n + 1) \text{ is true}]}_{\text{inductive conclusion}}$ then $P(n)$ is true for each $n \in \mathbb{Z}^{\geq n_0}$.**Ex3.** Let $\{a_n\}_{n=0}^{\infty}$ be the recursively defined sequence of integers

$$a_0 = 2 \quad , \quad a_1 = 4 \quad , \quad a_2 = 6$$

and

$$a_n = 5a_{n-3} \quad \text{when } n \in \mathbb{N} \text{ and } n \geq 3. \quad (\text{RD})$$

Prove that a_n is even for each $n \in \mathbb{Z}^{\geq 0} \stackrel{\text{i.e.}}{=} \{0, 1, 2, 3, 4, \dots\}$.►. Symbolically: $(\forall n \in \mathbb{Z}^{\geq 0}) \left[(a_0 = 2 \wedge a_1 = 4 \wedge a_2 = 6 \wedge (n \in \mathbb{N}^{\geq 3} \implies a_n = 5a_{n-3})) \implies a_n \text{ is even} \right]$ *Proof.* Let $\{a_n\}_{n=0}^{\infty}$ be the recursively defined sequence of integers

$$a_0 = 2 \quad , \quad a_1 = 4 \quad , \quad a_2 = 6$$

and

$$a_n = 5a_{n-3} \quad \text{when } n \in \mathbb{N} \text{ and } n \geq 3. \quad (\text{RD})$$

We will show that a_n is even for each $n \in \mathbb{Z}^{\geq 0}$ by strong induction on n .

For the base step, first let $n = 0$. Then $a_n = a_0 = 2$, which is even. Next let $n = 1$. Then $a_n = a_1 = 4$, which is even. Finally let $n = 2$. Then $a_n = a_2 = 6$, which is even. Thus a_0, a_1 , and a_2 are each even. This completes the base step.

For the inductive step, fix $n \in \mathbb{N}^{\geq 2}$ and assume the inductive hypothesis, which is

$$\text{if } j \in \{0, 1, 2, \dots, n\} \text{ then } a_j \text{ even.} \quad (\text{IH})$$

We will show the inductive conclusion, which is

$$a_{n+1} \text{ is even.} \quad (\text{IC})$$

Since $n \geq 2$,

$$n + 1 \geq 3$$

and so, by the recursive definition (RD) (the recursive definition has *kicked in* for a_{n+1} since $n + 1 \geq 3$)

$$a_{n+1} = 5a_{(n+1)-3}$$

and so

$$a_{n+1} = 5a_{n-2}. \quad (1)$$

Since $n \in \mathbb{Z}^{\geq 2}$, we know $2 \leq n$ and so

$$0 \leq n - 2 \leq n,$$

which gives $n - 2 \in \{0, 1, 2, \dots, n\}$. Thus we can apply the inductive hypothesis (IH) to $j = n - 2$ to get

$$a_{n-2} \text{ is even.} \quad (2)$$

Thus a_{n+1} is even by equations (1) and (2) and Lemma PEA. This completes the inductive step.Thus the base step and inductive step hold. So a_n is even for each $n \in \mathbb{Z}^{\geq 0}$. $\square$

Strong Induction (also called complete induction, our book calls this 2nd PMI)

Fix $n_0 \in \mathbb{Z}$.

If

BASE STEP: $P(n_0)$ is true

INDUCTIVE STEP: for each $n \in \mathbb{Z}^{\geq n_0}$: $\underbrace{[P(j) \text{ is true for } j \in \{n_0, 1 + n_0, \dots, n\}]}_{\text{inductive hypothesis}} \Rightarrow \underbrace{[P(n+1) \text{ is true}]}_{\text{inductive conclusion}}$

then $P(n)$ is true for each $n \in \mathbb{Z}^{\geq n_0}$.

Ex4. Theorem. Each natural number n has a factorization as

$$n = 2^k m$$

for some k is some nonnegative integer and some odd natural number m .

►. Written symbolically:

$$(\forall n \in \mathbb{N}) (\exists k \in \mathbb{Z}^{\geq 0}) (\exists m \in \mathbb{N}) [n = 2^k m \wedge m \text{ is odd}].$$

Thinking Land

Let's make a chart to help us understand better what is going on.

n	$n = 2^k m$ where $k \in \{0, 1, 2, 3, 4, 5, \dots\}$ and $m \in \{1, 3, 5, 7, 9, 11, \dots\}$
1	$1 = 2^0 \cdot 1$
2	$2 = 2^1 \cdot 1$
3	$3 = 2^0 \cdot 3$
4	$4 = 2^2 \cdot 1$
5	$5 = 2^0 \cdot 5$
6	$6 = 2^1 \cdot 3$
7	$7 = 2^0 \cdot 7$
So if n is <u>odd</u> , then $n = 2^0 \cdot n$ so $k := 0 \in \mathbb{Z}^{\geq 0}$ and $m := n$ with m an odd natural number.	
8	$8 = 2^1 \cdot 4 = 2^1 \cdot (2^{k_4} \cdot m_4) = 2^{1+k_4} \cdot m_4$
10	$10 = 2^1 \cdot 5 = 2^1 \cdot (2^{k_5} \cdot m_5) = 2^{1+k_5} \cdot m_5$
12	$12 = 2^1 \cdot 6 = 2^1 \cdot (2^{k_6} \cdot m_6) = 2^{1+k_6} \cdot m_6$

◦. For the Base Step, which n 's do we need to check? $n = 1$.

◦. Since in the Base Step we verified the Thm. holds up to (and including) $n = \underline{1}$, where should we start the Induction Step? At $n = \underline{1}$.

So the first line in your induction step should look something like:

For the inductive step, fix $n \in \mathbb{N}$ such that $n \geq \underline{1}$. Assume the inductive hypothesis, which is

$$\text{if } j \in \{1, 2, \dots, n\} \text{ then } j = 2^{k_j} m_j \text{ for some } k_j \in \mathbb{Z}^{\geq 0} \text{ and odd natural number } m_j. \quad (\text{IH})$$

We will show the inductive conclusion, which is

$$n + 1 = 2^k m \quad \text{for some } k \in \mathbb{Z}^{\geq 0} \text{ and odd natural number } m. \quad (\text{IC})$$

To show the (IC), we will need to consider (the only possible) two cases for n : n is even and n is odd.

Ex4. Written symbolically: $(\forall n \in \mathbb{N}) (\exists k \in \mathbb{Z}^{\geq 0}) (\exists m \in \mathbb{N}) [n = 2^k m \wedge m \text{ is odd}]$.

Proof. We shall show that if $n \in \mathbb{N}$ then n can be written as

$$n = 2^k m \quad \text{for some } k \in \mathbb{Z}^{\geq 0} \text{ and odd natural number } m. \quad (1)$$

by strong induction on n .

For the base step, let $n = 1$. Then

$$n = 1 = 2^0 \cdot 1 = 2^k m$$

where $k = 0 \in \mathbb{Z}^{\geq 0}$ and $m = 1 \in \mathbb{N}$ is odd. So (1) holds when $n = 1$. This completes the base step.

For the inductive step, fix $n \in \mathbb{N}$. Assume the inductive hypothesis, which is

$$\text{if } j \in \{1, 2, \dots, n\} \text{ then} \quad (\text{IH})$$

$$j = 2^{k_j} m_j \quad \text{for some } k_j \in \mathbb{Z}^{\geq 0} \text{ and odd natural number } m_j.$$

We will show the inductive conclusion, which is

$$n + 1 = 2^k m \quad \text{for some } k \in \mathbb{Z}^{\geq 0} \text{ and odd natural number } m, \quad (\text{IC})$$

by considering (the only possible) two cases: n is even and n is odd.

For the first case, let n be an even natural number. Then $n + 1$ is an odd natural number so

$$n + 1 = 2^0 (n + 1) = 2^k m$$

where $k = 0 \in \mathbb{Z}^{\geq 0}$ and $m = n + 1$ is an odd natural number. Thus (IC) holds for the first case.

For the second case, let n be an odd natural number. Then $n + 1$ is an even natural number; thus, there is $l \in \mathbb{N}$ such that

$$n + 1 = 2l. \quad (2)$$

Note that $l \in \{1, 2, \dots, n\}$ since $l \in \mathbb{N}$ and

$$1 \leq l = \frac{n + 1}{2} \leq \frac{n + n}{2} = n.$$

Thus by the inductive hypotheses (IH) (think $j := l$) there exists $k_l \in \mathbb{Z}^{\geq 0}$ and an odd natural number m_l such that

$$l = 2^{k_l} m_l. \quad (3)$$

Equations (2) and (3) give,

$$n + 1 = 2l = 2(2^{k_l} m_l) = (2^{1+k_l}) (m_l) = 2^k m$$

where $m := m_l$ is an odd natural number and $k := 1 + k_l \in \mathbb{Z}^{\geq 0}$ (since $k_l \in \mathbb{Z}^{\geq 0}$). Thus (IC) holds for the second case. This completes the inductive step.

Thus the base step and inductive step hold. So (1) holds for all $n \in \mathbb{N}$ by strong induction. $\square$