Ex1. Prove that $\displaystyle\sum_{i=1}^{n} \frac{1}{i^2} \le 2 - \frac{1}{n}$ for each integer n.

WTS. $(\forall n \in \mathbb{N})\,[P(n) \text{ is true}]$ where $P(n)$ is the open sentence $\sum_{i=1}^{n} \frac{1}{i^2} \le 2 - \frac{1}{n}$ in the variable $n \in \mathbb{N}$.

Proof. Using basic induction on the variable n, we will show that for each $n \in \mathbb{N}$

$$\sum_{i=1}^{n} \frac{1}{i^2} \le 2 - \frac{1}{n}. \tag{1}$$

For the base step, let $n = 1$. Since, when $n = 1$,

$$\sum_{i=1}^{n} \frac{1}{i^2} = \sum_{i=1}^{1} \frac{1}{i^2} = \frac{1}{1^2} = 1 \qquad \text{and} \qquad 2 - \frac{1}{n} = 2 - \frac{1}{1} = 2 - 1 = 1,$$

inequality (1) holds when $n = 1$. This finishes the base step.

For the inductive step, fix $n \in \mathbb{N}$. We assume the inductive hypothesis, which is $\langle\, P(n) \text{ is true} \,\rangle$

$$\sum_{i=1}^{n} \frac{1}{i^2} \le 2 - \frac{1}{n}. \tag{IH}$$

For the inductive step, your goal is to show the inductive conclusion, which is $\langle\, P(n+1) \text{ is true} \,\rangle$

$$\sum_{i=1}^{n+1} \frac{1}{i^2} \;\le\; 2 - \frac{1}{n+1}. \tag{IC}$$

We now have $\langle$ recall $\displaystyle\sum_{i=1}^{n+1} a_i = (a_1 + a_2 + \cdots + a_n) + a_{n+1} = \left(\sum_{i=1}^{n} a_i\right) + a_{n+1}$ $\rangle$

$$\sum_{i=1}^{n+1} \frac{1}{i^2} = \left(\sum_{i=1}^{n} \frac{1}{i^2}\right) + \frac{1}{(n+1)^2}$$

and by the inductive hypothesis (IH)

$$\le \left(2 - \frac{1}{n}\right) + \frac{1}{(n+1)^2}$$

$\langle$whenever do not know what to do next, LOOK at (IC) for hint on where to go next$\rangle$

$$=$$

$$=$$

$$=$$

$$=$$

(inequality help: $n^2 + n + 1 \;\square\; n^2 + n$ so $\frac{n^2+n+1}{n^2+n} \;\square\; \frac{n^2+n}{n^2+n}$ so $-\left(\frac{1}{n+1}\right) \frac{n^2+n+1}{n^2+n} \;\square\; -\left(\frac{1}{n+1}\right) \frac{n^2+n}{n^2+n}$)

$$\le$$

$$= 2 - \left(\frac{1}{n+1}\right).$$

Thus (IC) hold. This completes the inductive step.

Thus, by induction, (1) holds for each $n \in \mathbb{N}$. $\square$

Rmk. When we write an induction proof, we usally write the Base Step first.
However, in your *Thinking Land*, we usually do the Inductive Step first. Why?
Let's say we want to show a $\left(\forall n \in \mathbb{Z}^{\geq 5}\right)[\,P(n)\,]$ and our inductive step (i.e., $P(n) \implies P(n+1)$) only works when $n \geq 7$ (and our inductive step just does not work when n is 5 or 6). All is not lost! In this situation, we need to show the base step $P(n)$ hold true when n is: ____________.

Ex2. Prove that for $n \in \mathbb{N}$ with $n \geq 6$

$$n^3 < n!\ .$$

Proof. We shall show that for each $n \in \mathbb{N}^{\geq 6}$

$$n^3 < n! \tag{1}$$

by ⟨extended/generalized⟩ induction on n.

For the base step, let $n = 6$. Then

$$n^3 = 6^3 = 216. \tag{2}$$

while

$$n! = 6! = 720. \tag{3}$$

Since $216 < 720$, the inequality in (1) holds when $n = 6$. This completes the base step.

For the inductive step, fix a natural number $n \in \mathbb{N}^{\geq 6}$. Assume that

$$n^3 < n!. \tag{IH}$$

We need to show that

$$(n+1)^3 < (n+1)!. \tag{IC}$$

We now compute:

$$(n+1)! \;=\; (n+1)\ [\ n!\ ]$$

and by the inductive hypotheses (IH)

$$> (n+1)\ \left[\ n^3\ \right]$$

⟨Look at (IC), which holds if $n^3 \overset{\text{go for}}{\geq} (n+1)^2$. Since $6 \leq n$, we know $(n+1)^2 \leq (n+n)^2 = (2n)^2 = 4n^2 \leq 6n^2 \leq n \cdot n^2 = n^3$.⟩

$$= (n+1)\ n \cdot n^2$$

and since $n \geq 6 \geq 4$

$$\geq$$

$$=$$

$$=$$

and since $n \in \mathbb{N}$ so $n \geq 1$

$$\geq\ (n+1)\ (n+1)^2$$

$$= (n+1)^3\ .$$

Thus inequality (IC) hold. This completes the inductive step.

Thus, by induction, inequality (1) holds for each natural number $n \in \mathbb{N}^{\geq 6}$. □☺☺.

Strong Induction (also called complete induction, our book calls this 2[nd] PMI) §4.2 p194

Fix $n_0 \in \mathbb{Z}$.
If

BASE STEP: $P(n_0)$ is true

INDUCTIVE STEP: for each $n \in \mathbb{Z}^{\geq n_0}$: $\underbrace{[\ P(j) \text{ is true for } j \in \{n_0, 1+n_0, \ldots, n\}\ ]}_{\text{inductive hypothesis}} \Rightarrow \underbrace{[\ P(n+1) \text{ is true}\ ]}_{\text{inductive conclusion}}$

then $P(n)$ is true for each $n \in \mathbb{Z}^{\geq n_0}$.

Ex3. Let $\{a_n\}_{n=0}^{\infty}$ be the recursively defined sequence of integers

$$a_0 = 2 \quad , \quad a_1 = 4 \quad , \quad a_2 = 6$$

and

$$a_n = 5a_{n-3} \quad \text{when} \quad n \in \mathbb{N} \text{ and } n \geq 3. \tag{RD}$$

Prove that a_n is even for each $n \in \mathbb{Z}^{\geq 0} \overset{\text{i.e.}}{=} \{0, 1, 2, 3, 4, \ldots\}$. RD = Recursive Def. ↑

▸. Symbolically:

Thinking Land

Let's make a chart to help us understand better what is going on.

n	a_n
0	$a_0 = 2$ (given)
1	$a_1 = 4$ (given)
2	$a_2 = 6$ (given)
	now the recursive definition *kicks in* that $a_n = 5a_{n-3}$
3	$a_3 =$
4	$a_4 =$
5	$a_5 =$
6	$a_6 =$
7	$a_7 =$
8	$a_8 =$
	Do we see a pattern?

◦. For the Base Step, which n's do we need to check? ______________________________.

◦. Since in the Base Step we verified the Thm. holds up to (and including) $n =$ ____,

where should we start the Induction Step? At $n =$ ____.

So the first line in your induction step should look something line:

For the inductive step, fix $n \in \mathbb{N}$ such that $n \geq$ ____. Assume the inductive hypothesis, which is

(IH)

We will show the inductive conclusion, which is

(IC)

Strong Induction. §4.2 p194

Fix $n_0 \in \mathbb{Z}$.

If

BASE STEP: $P(n_0)$ is true

INDUCTIVE STEP: for each $n \in \mathbb{Z}^{\geq n_0}$: $\underbrace{[\ P(j) \text{ is true for } j \in \{n_0, 1+n_0, \ldots, n\}\ ]}_{\text{inductive hypothesis}} \Rightarrow \underbrace{[\ P(n+1) \text{ is true}\ ]}_{\text{inductive conclusion}}$

then $P(n)$ is true for each $n \in \mathbb{Z}^{\geq n_0}$.

Ex3. Let $\{a_n\}_{n=0}^{\infty}$ be the recursively defined sequence of integers

$$a_0 = 2 \quad , \quad a_1 = 4 \quad , \quad a_2 = 6$$

and

$$a_n = 5a_{n-3} \quad \text{when} \quad n \in \mathbb{N} \text{ and } n \geq 3. \tag{RD}$$

Prove that a_n is even for each $n \in \mathbb{Z}^{\geq 0} \overset{\text{i.e.}}{=} \{0, 1, 2, 3, 4, \ldots\}$.

►. Symbolically: $(\forall n \in \mathbb{Z}^{\geq 0})\ \left[\ \left(a_0 = 2 \wedge a_1 = 4 \wedge a_2 = 6 \wedge \left(n \in \mathbb{N}^{\geq 3} \implies a_n = 5a_{n-3}\right)\right) \implies a_n \text{ is even}\ \right]$

Proof. Let $\{a_n\}_{n=0}^{\infty}$ be the recursively defined sequence of integers

$$a_0 = 2 \quad , \quad a_1 = 4 \quad , \quad a_2 = 6$$

and

$$a_n = 5a_{n-3} \quad \text{when} \quad n \in \mathbb{N} \text{ and } n \geq 3. \tag{RD}$$

We will show that a_n is even for each $n \in \mathbb{Z}^{\geq 0}$ by strong induction on n.

For the base step, first let $n = 0$. Then $a_n = a_0 = 2$, which is even. Next let $n = 1$ Then $a_n = a_1 = 4$, which is even. Finally let $n = 2$. Then $a_n = a_2 = 6$, which is even. Thus a_0, a_1, and a_2 are each even. This completes the base step.

For the inductive step, fix $n \in \mathbb{N}^{\geq 2}$ and assume the inductive hypothesis, which is

$$\text{if } j \in \{0, 1, 2, \ldots, n\} \text{ then } a_j \text{ even.} \tag{IH}$$

We will show the inductive conclusion, which is

$$a_{n+1} \text{ is even.} \tag{IC}$$

Since $n \geq 2$,

$$n + 1 \geq 3$$

and so, by the recursive definition (RD) ⟨the recurive definition has *kicked in* for a_{n+1} since $n + 1 \geq 3$⟩

$$a_{n+1} = 5a_{(n+1)-3}$$

and so

$$a_{n+1} = 5a_{n-2}. \tag{1}$$

Since $n \in \mathbb{Z}^{\geq 2}$, we know $2 \leq n$ and so

$$0 \leq n - 2 \leq n,$$

which gives $n - 2 \in \{0, 1, 2, \ldots, n\}$. Thus we can apply the inductive hypothesis (IH) to $j = n - 2$ to get

$$a_{n-2} \text{ is even.} \tag{2}$$

Thus a_{n+1} is even by equations (1) and (2) and Lemma PEA. This completes the inductive step.

Thus the base step and inductive step hold. So a_n is even for each $n \in \mathbb{Z}^{\geq 0}$. □

Strong Induction (also called complete induction, our book calls this 2nd PMI) §4.2
Fix $n_0 \in \mathbb{Z}$. p194
If

BASE STEP: $P(n_0)$ is true
INDUCTIVE STEP: for each $n \in \mathbb{Z}^{\geq n_0}$: $\underbrace{[\ P(j) \text{ is true for } j \in \{n_0, 1+n_0, \ldots, n\}\ ]}_{\text{inductive hypothesis}} \Rightarrow \underbrace{[\ P(n+1) \text{ is true}\ ]}_{\text{inductive conclusion}}$

then $P(n)$ is true for each $n \in \mathbb{Z}^{\geq n_0}$.

Ex4. **Theorem**. Each natural number n has a factorization as

$$n = 2^k m$$

for some k is some nonnegative integer and some odd natural number m.

►. Written symbolically:

Thinking Land

Let's make a chart to help us understand better what is going on.

n	$n = 2^k m$ where $k \in \{0, 1, 2, 3, 4, 5, \ldots\}$ and $m \in \{1, 3, 5, 7, 9, 11, \ldots\}$
1	$1 =$
2	$2 =$
3	$3 =$
4	$4 =$
5	$5 =$
6	$6 =$
7	$7 =$
So if n is _____, then $n =$ _____ so $k :=$__ $\in \mathbb{Z}^{\geq 0}$ and $m :=$ __ with m an odd natural number.	
8	$8 =$
10	$10 =$
12	$12 =$

◦. For the Base Step, which n's do we need to check? ________________________.

◦. Since in the Base Step we verified the Thm. holds up to (and including) $n =$ ____,

where should we start the Induction Step? At $n =$ ____.

So the first line in your induction step should look something line:

For the inductive step, fix $n \in \mathbb{N}$ such that $n \geq$ ____. Assume the inductive hypothesis, which is

(IH)

We will show the inductive conclusion, which is

(IC)

To show the (IC), we will need to consider ⟨the only possible⟩ two cases for n: _________ and _________.

Ex4. Written symbolically: $(\forall n \in \mathbb{N}) \left(\exists k \in \mathbb{Z}^{\geq 0}\right) (\exists m \in \mathbb{N}) \left[n = 2^k m \ \wedge\ m \text{ is odd } \right].$

Proof. We shall show that if $n \in \mathbb{N}$ then n can be written as

$$n = 2^k m \qquad \text{for some } k \in \mathbb{Z}^{\geq 0} \text{ and odd natural number } m. \tag{1}$$

by strong induction on n.

For the base step, let $n = 1$. Then

$$n = 1 = 2^0 \cdot 1 = 2^k m$$

where $k = 0 \in \mathbb{Z}^{\geq 0}$ and $m = 1 \in \mathbb{N}$ is odd. So (1) holds when $n = 1$. This completes the base step.

For the inductive step, fix $n \in \mathbb{N}$. Assume the inductive hypothesis, which is

$$\begin{gathered} \text{if } j \in \{1, 2, \dots, n\} \text{ then} \\ j = 2^{k_j} m_j \quad \text{for some } k_j \in \mathbb{Z}^{\geq 0} \text{ and odd natural number } m_j. \end{gathered} \tag{IH}$$

We will show the inductive conclusion, which is

$$n + 1 = 2^k m \qquad \text{for some } k \in \mathbb{Z}^{\geq 0} \text{ and odd natural number } m, \tag{IC}$$

by considering ⟨the only possible⟩ two cases: n is even and n is odd.

For the first case, let n be an even natural number. Then $n+1$ is an odd natural numbers so

$$n + 1 = 2^0 \, (n+1) = 2^k m$$

where $k = 0 \in \mathbb{Z}^{\geq 0}$ and $m = n + 1$ is an odd natural number. Thus (IC) holds for the first case.

For the second case, let n be an odd natural number. Then $n+1$ is an even natural number; thus, there is $l \in \mathbb{N}$ such that

$$n + 1 = 2l. \tag{2}$$

Note that $l \in \{1, 2, \dots, n\}$ since $l \in \mathbb{N}$ and

$$1 \leq l = \frac{n+1}{2} \leq \frac{n+n}{2} = n.$$

Thus by the inductive hypotheses (IH) ⟨think $j := l$⟩ there exists $k_l \in \mathbb{Z}^{\geq 0}$ and an odd natural number m_l such that

$$l = 2^{k_l} m_l. \tag{3}$$

Equations (2) and (3) give,

$$n + 1 = 2l = 2\left(2^{k_l} m_l\right) = \left(2^{1+k_l}\right) (m_l) = 2^k m$$

where $m{:=}m_l$ is an odd natural number and $k{:=}1 + k_l \in \mathbb{Z}^{\geq 0}$ (since $k_l \in \mathbb{Z}^{\geq 0}$). Thus (IC) holds for the second case. This completes the inductive step.

Thus the base step and inductive step hold. So (1) holds for all $n \in \mathbb{N}$ by strong induction. □