1.6. Section 1.6: Proofs Involving Quantifiers.

1.6.1. Remark. Many theorems in mathematics are quantified sentences, even though the quantifier may not actually appear in the statement. For example, proving "If $x$ is an odd integer, then $x + 1$ is even," as we did in Section 1.4, actually involves a quantifier, since the sentence has the symbolic translation

$$(\forall x \in \mathbb{Z}) (x \text{ is odd } \Rightarrow x + 1 \text{ is even}).$$

In Sections 1.4 and 1.5, in order to concentrate on the basic proof forms, we suppressed the importance of quantifiers by viewing the variable as a fixed object; thus, our proofs of quantified statements in Sections 1.4 and 1.5 were not correct. In this section, we will learn how to correctly prove quantified sentences.

---

1.6.2. Proof of $(\forall x)P(x)$ (Direct Proof).

Let $x$ be an arbitrary object in the universe.
(or ... Fix an arbitrary element $x$ from the universe.)

Hence, $P(x)$ is true.
Since $x$ is arbitrary, $(\forall x)P(x)$ is true.

1.6.3. Proof of $(\forall x)P(x)$ (Proof by Contradiction).

Assume $\sim [(\forall x)P(x)]$ is true.
Then $(\exists x)[\sim P(x)]$ is true.
Let $t$ be an object such that $\sim P(t)$ is true.

Therefore $Q \land \sim Q$ is true, a contradiction.
Hence, $\sim [(\forall x)P(x)]$ is false;
so its denial $(\forall x)P(x)$ is true.

---

1.6.4. Proof of $(\exists x)P(x)$ (Constructive (direct) Proof).

Specifically describe some object $x$ in the universe that makes $P(x)$ true.

1.6.5. Proof of $(\exists x)P(x)$ (Nonconstructive (indirect) Proof).

Show that there must be some object $x$ in the universe that makes $P(x)$ true,
without ever actually producing the object.

1.6.6. Proof of $(\exists x)P(x)$ (Proof by Contradiction).

Assume $\sim [(\exists x)P(x)]$ is true.
Thus $(\forall x)[\sim P(x)]$ is true.

Therefore $Q \land \sim Q$ is true, a contradiction.
Hence, $\sim (\exists x)P(x)$ is false;
so $(\exists x)P(x)$ is true.
1.6.7. **Proof of** \( (\exists x)P(x) \).

Do in two parts:

1\textsuperscript{st} *Existence*:
Prove that \( (\exists x)P(x) \) is true by any method.

2\textsuperscript{nd} *Uniqueness*:
Assume that \( t_1 \) and \( t_2 \) are objects in the universe such that \( P(t_1) \) and \( P(t_2) \) are true.

\[ \therefore \]

Therefore, \( t_1 = t_2 \).

We conclude that \( (\exists x)P(x) \).

---

1.6.8. **Definition.** A *counterexample* to \( (\forall x)P(x) \) is any object \( t \) in the universe for which \( P(t) \) is false.

---

1.6.9. **Remark** (Some helpful **TRUE** propositions.).

\[
\begin{align*}
\star 1. \quad (\forall x)(\forall y)P(x, y) & \iff (\forall y)(\forall x)P(x, y) \\
\star 2. \quad (\exists x)(\exists y)P(x, y) & \iff (\exists y)(\exists x)P(x, y) \\
\star 3. \quad (\forall x)[ P(x) \land Q(x) ] & \iff [ (\forall x)P(x) \land (\forall x)Q(x) ] \\
\star 4. \quad [ (\forall x)P(x) \lor (\forall x)Q(x) ] & \implies (\forall x) [ P(x) \lor Q(x) ] \\
\star 5. \quad (\forall x)[ P(x) \implies Q(x) ] & \implies [ (\forall x)P(x) \implies (\forall x)Q(x) ] \\
\star 6. \quad (\exists x)(\forall y)P(x, y) & \implies (\forall y)(\exists x)P(x, y)
\end{align*}
\]

1.6.10. **Remark** (Some **FALSE** propositions that are common errors.).

\[
\begin{align*}
\star 7. \quad (\exists x)P(x) & \implies (\forall x)P(x) \\
\star 8. \quad (\forall x)[ P(x) \lor Q(x) ] & \implies [ (\forall x)P(x) \lor (\forall x)Q(x) ] \quad \text{converse of 4} \\
\star 9. \quad [ (\forall x)P(x) \implies (\forall x)Q(x) ] & \implies (\forall x) [ P(x) \implies Q(x) ] \quad \text{converse of 5} \\
\star 10. \quad (\forall y)(\exists x)P(x, y) & \implies (\exists x)(\forall y)P(x, y) \quad \text{converse of 6}
\end{align*}
\]

- To help remember 7., 8., and 9., let \( U = \mathbb{N} \), \( P(x) \): \( x \) is even, \( Q(x) \): \( x \) is odd.
- To help remember 10., let \( U = \mathbb{R}^2 \), \( P(x, y) \): \( x < y \).