Remark. Think about taking an \( n \in \mathbb{N} \) and dividing it by \( d \in \mathbb{N} \). What happens?

Let’s look at an example: take \( n = 11 \) and divide it by \( d = 5 \) to get

\[
\frac{11}{5} = 2 \frac{1}{5} \quad \text{or equivalently} \quad 11 = 5 \times 2 + 1 .
\]

In general, one can divide \( n \in \mathbb{N} \) by \( d \in \mathbb{N} \). Think of as:

\[
d \sqrt[n]{n} \quad \vdots \quad r
\]

to get

\[
\frac{n}{d} = q \frac{r}{d} \quad \text{or equivalently} \quad n = dq + r
\]

for some quotient \( q \in \mathbb{N} \cup \{0\} \) and some remainder \( r \in \mathbb{N} \cup \{0\} \) where \( 0 \leq r < d \).

**Theorem.** Division Algorithm for \( \mathbb{N} \)

\[
( \forall n \in \mathbb{N} ) \ ( \forall d \in \mathbb{N} ) \ ( \exists ! q \in \mathbb{N} \cup \{0\} ) \ ( \exists ! r \in \mathbb{N} \cup \{0\} ) \ [ (n = dq + r) \land (0 \leq r < d) ]
\]

equivalently

\[
( \forall n \in \mathbb{N} ) \ ( \forall d \in \mathbb{N} ) \ ( \exists ! q \in \mathbb{N} \cup \{0\} ) \ ( \exists ! r \in \{0,1,\ldots,d-1\} ) \ [ n = dq + r ]
\]

Remark. Here: \( r \in \{0,1,2,\ldots,d-1\} \) so there are \( d \) possibilities for \( r \).

**Theorem.** Division Algorithm for \( \mathbb{Z} \). (7th edition, page 62)

\[
( \forall n \in \mathbb{Z} ) \ ( \forall d \in \mathbb{Z} \setminus \{0\} ) \ ( \exists ! q \in \mathbb{Z} ) \ ( \exists ! r \in \mathbb{Z} ) \ [ (n = dq + r) \land (0 \leq r < |d|) ]
\]

equivalently

\[
( \forall n \in \mathbb{Z} ) \ ( \forall d \in \mathbb{Z} \setminus \{0\} ) \ ( \exists ! q \in \mathbb{Z} ) \ ( \exists ! r \in \{0,1,\ldots,|d|-1\} ) \ [ n = dq + r ]
\]

Remark. Here: \( r \in \{0,1,2,\ldots,(|d|-1)\} \) so there are \( d \) possibilities for \( r \).