INSTRUCTIONS:

(1) The **MARK BOX** indicates the problems along with their points.
    Check that your copy of the exam has all of the problems.

(2) When applicable put your answer on/in the line/box provided.
    Show your work **UNDER** the provided line/box.
    If no such line/box is provided, then box your answer.

(3) For problems that request a proof, write a neat **FORMAL** proof and use only logic and the
    definitions of the concepts involved (i.e., do not quote a problem from the book). You may
    include your skeleton, if you so wish.

(4) During this exam, do not leave your seat. If you have a question, raise your hand. When
    you finish: turn your exam over, put your pencil down, and raise your hand.

(5) You may **not** use a calculator, books, personal notes.

(6) This exam covers (from *A Transition to Advanced Mathematics* by Smith, Eggen, and St.
    Andre: 6th ed.): Sections 1.1 – 1.7 .

**Numbers:**

- real numbers = \( \mathbb{R} \)
- **positive** real numbers = \( \mathbb{R}^+ = \{x \in \mathbb{R}: x > 0\} \)
- natural numbers = \( \mathbb{N} = \{1, 2, 3, 4, \ldots\} \)
- integers = \( \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\} \)
- rational numbers = \( \mathbb{Q} = \{\frac{p}{q} \in \mathbb{R}: p, q \in \mathbb{Z} \text{ and } q \neq 0\} \)

**Throughout this exam:**

(1) \( P, Q, \) and \( R \) be propositions.

(2) \( P(x_1, \ldots, x_n), Q(x_1, \ldots, x_n), \) and \( R(x_1, \ldots, x_n) \) are open sentences with variables \( x_1, \ldots, x_n. \)
1. Make true statements by filling in the blanks with the appropriate number between 1 and 11 from below. Use each number once, and only once.

\[
\begin{align*}
(1) & \quad (P \implies Q) \land (P \implies R) \\
(2) & \quad (P \implies Q) \land (Q \implies P) \\
(3) & \quad (P \implies R) \land (Q \implies R) \\
(4) & \quad (P \land \sim Q) \implies R \\
(5) & \quad P \implies \sim Q \\
(6) & \quad \sim Q \implies \sim P \\
(7) & \quad (\sim P) \land (\sim Q) \\
(8) & \quad (P \land \sim Q) \\
(9) & \quad (\forall x)[\sim P(x)] \\
(10) & \quad (\exists x)[\sim P(x)] \\
(11) & \quad (\exists x)[P(x) \land (\forall y)(P(y) \implies x = y)]
\end{align*}
\]

<table>
<thead>
<tr>
<th>Statement</th>
<th>Equivalent To</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P \implies Q)</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>(P \iff Q)</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>(\sim (P \implies Q))</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>(\sim (P \land Q))</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>(\sim (P \lor Q))</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>(P \implies (Q \land R))</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>(P \implies (Q \lor R))</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>((P \lor Q) \implies R)</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>(\sim (\forall x)P(x))</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>(\sim (\exists x)P(x))</td>
<td></td>
<td>9</td>
</tr>
<tr>
<td>((\exists x)!P(x))</td>
<td></td>
<td>11</td>
</tr>
</tbody>
</table>
2. Circle T if and only if the statement is TRUE. Circle F if and only if the statement is FALSE.

1. T F \((\forall x)(\forall y)P(x,y) \implies (\forall y)(\forall x)P(x,y)\)

2. F F \((\exists x)(\exists y)P(x,y) \implies (\exists y)(\exists x)P(x,y)\)

3. T F \((\forall x)P(x) \implies (\exists x)P(x)\)

4. T F \((\exists x)P(x) \implies (\forall x)P(x)\)

5. \(\forall \) F \((\forall x)[P(x) \lor Q(x)] \implies [(\forall x)P(x) \lor (\forall x)Q(x)]\)

6. \(\forall \) F \([(\forall x)P(x) \lor (\forall x)Q(x)] \implies (\forall x)[P(x) \lor Q(x)]\)

7. \(\forall \) F \((\forall x)[P(x) \implies Q(x)] \implies [(\forall x)P(x) \implies (\forall x)Q(x)]\)

8. \(\forall \) F \[(\forall x)[P(x) \implies Q(x)] \implies (\forall x)[P(x) \implies Q(x)]\)

9. \(\forall \) F \[(\forall x)P(x) \implies (\forall x)Q(x)] \implies (\forall x)[P(x) \implies Q(x)]\)

10. \(\forall \) F \((\forall y)(\exists x)P(x,y) \implies (\exists x)(\forall y)P(x,y)\)

11. \(\forall \) F \((\exists x)(\forall y)P(x,y) \implies (\forall y)(\exists x)P(x,y)\)

5. \(\forall \) F \((\forall x)[P(x) \land Q(x)] \implies [(\forall x)P(x) \land (\forall x)Q(x)]\)

5. \(\forall \) F \([(\forall x)P(x) \land (\forall x)Q(x)] \implies (\forall x)[P(x) \land Q(x)]\)
3. Do THREE problems from problems: 3a, 3b, 3c, and 3d. Fill-in the blanks on this sheet of paper but include your proof parts on a separate sheet of paper (with your name on each sheet please).

Circle THREE: I am doing problems 3a, 3b, 3c, 3d

3a. Let \( x \in \mathbb{Z} \). If 8 does not divide \( x^2 - 1 \), then \( x \) is even.

- Let \( a, b, c \in \mathbb{Z} \). By definition:
  
  \( a \) divides \( b \), denoted by \( a \mid b \), if and only if \( (\exists k \in \mathbb{Z}) \ (a \cdot k = b) \)
  
  \( c \) is even if and only if \( (\exists k \in \mathbb{Z}) \ (c = 2k) \)
  
  \( c \) is odd if and only if \( (\exists k \in \mathbb{Z}) \ (c = 2k - 1) \)

- Symbolically write, using quantifiers, statement 3a.
  
  \((\forall x \in \mathbb{Z})(8 \mid x^2 - 1 \Rightarrow x \text{ is even})\)

- Prove statement 3a.

3b. There exists an odd integer \( M \) such that for each real number \( r \) larger than \( M \), \( \frac{1}{2r} < 0.001 \).

- By definition, an integer \( M \) is odd if and only if \( (\exists k \in \mathbb{Z}) \ (M = 2k - 1) \)

- Symbolically write, using quantifiers, statement 3b.
  
  \((\exists M \in \mathbb{Z} - 1)(\forall r \in \mathbb{R}) \ (r > M \Rightarrow \frac{1}{2r} < 0.001)\)

- Prove statement 3b.
3c. For each rational number $x$ and each irrational number $y$, there is a unique irrational number $z$ such that $y + z = x$.

- Let $a$ and $b$ be real numbers. By definition:

  $a$ is rational if and only if $(\exists p \in \mathbb{Z})(\exists q \in \mathbb{Z})(a = \frac{p}{q} \land q \neq 0)$

  $b$ is irrational if and only if $b \notin \mathbb{Q}$, i.e., $b \in \mathbb{R} \setminus \mathbb{Q}$

- Symbolically write, using quantifiers, statement 3c.

  $$(\forall x \in \mathbb{Q})(\forall y \in \mathbb{R} \setminus \mathbb{Q})(\exists ! z \in \mathbb{R} \setminus \mathbb{Q})(y + z = x)$$

- Prove statement 3c. You must carefully prove, using only the definitions above (and not using another homework problem), that your $z$ is indeed irrational.

3d. Let $a, b, c, d \in \mathbb{N}$ and $d = \text{GCD}(a, b)$. If $c$ divides $a$ and $c$ divides $b$, then $c$ divides $d$.

- By definition:

  $c$ divides $d$, denoted $c | d$, if and only if $(\exists k \in \mathbb{N})(c \cdot k = d)$

  $d$ is the greatest common divisor of $a$ and $b$, denoted by $d = \text{GCD}(a, b)$, if and only if

  1. $d | a$ and $d | b$ i.e., $d$ is a common divisor of $a$ and $b$.
  2. if $c | a$ and $c | b$ then $c \leq d$.

- Symbolically write, using quantifiers, statement 3d.

  $$(\forall (a, b, c) \in \mathbb{N}^3)(c | a \land c | b \Rightarrow c | \text{GCD}(a, b))$$

- Prove statement 3d.
(∀x ∈ ℤ) \( \frac{8 | x^2 - 1 \Rightarrow x \text{ is even}}{P(x)} \).}

**Skeleton**

Fix \( x ∈ ℤ \).

Assume \( x \) is odd

WTS \( \frac{P(x) \Rightarrow Q(x)}{\text{i.e. WTS } 8 | x^2 - 1} \).

\( x \text{ odd} \Rightarrow (x = 2k + 1) \)

So \( x^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k(k + 1) \), even so \( 2j = 8j \).

**Formal**

Fix \( x ∈ ℤ \).

Let \( x \) be odd. Then \( x = 2k + 1 \) for some \( k ∈ ℤ \). Thus

\[ x^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k(k + 1). \]

Note that \( k(k + 1) \) is even so \( k(k + 1) = 2j \) for some \( j ∈ ℤ \). So

\[ x^2 - 1 = 4k(k + 1) = 4 \cdot 2j = 8j. \]

So by definition of divides, \( 8 | (x^2 - 1) \).

Thus if \( x \) is odd then \( 8 | (x^2 - 1) \). So if \( 8 \) does not divide \( x^2 - 1 \), then \( x \) is even.
\[
\exists m \in \mathbb{Z} - 1 \land (\forall r \in \mathbb{R}) \quad (r > M \Rightarrow \frac{1}{2r} < 0.001)
\]

Skeletion
\[
\Leftrightarrow \exists m \in \mathbb{Z} - 1 \land (\forall r \in \mathbb{R}) \quad (M < r \Rightarrow \frac{1}{2r} < \frac{1}{1000})
\]
\[
\Leftrightarrow \exists m \in \mathbb{Z} - 1 \land (\forall r \in \mathbb{R}) \quad (M < r \Rightarrow \frac{1000}{2r} < r)
\]
\[
\Leftrightarrow \exists m \in \mathbb{Z} - 1 \land (\forall r \in \mathbb{R}) \quad (M < r \Rightarrow 500 < r).
\]

**Formal.** Let \( M = 501 \). Clearly \( M \) is an odd integer. Fix \( r \in \mathbb{R} \).

If \( r > M \), i.e. if

\[
r > 501
\]

then by (i)

\[
\frac{1}{2r} < \frac{1}{2(501)} = \frac{1}{1002} < \frac{1}{1000} = 0.001.
\]
\[(\forall x \in \mathbb{Q}) (\forall y \in \mathbb{R} \setminus \mathbb{Q}) (\exists! z \in \mathbb{R} \setminus \mathbb{Q}) (y + z = x).\]

**Skeleton**

Fix \( x \in \mathbb{Q} \) \( \neq y \in \mathbb{Q}. \)

WITF \( z \) s.t. \( y + z = x \) \( \iff \)

\[ z = x - y \]

**Formal**

Fix \( x \in \mathbb{Q} \) and \( y \in \mathbb{R} \setminus \mathbb{Q}. \)

**Existence**

Let \( z = x - y. \) Clearly \( y + z = x. \) Next we want to show that \( z \notin \mathbb{Q}. \) So assume \( z \in \mathbb{Q}. \) \( \iff \)

Since \( x, z \in \mathbb{Q} \)

\[ x = \frac{p}{q} \quad \text{and} \quad z = \frac{s}{t} \]

for some \( p, q, s, t \in \mathbb{Z} \) with \( q \neq 0 \) and \( t \neq 0. \) Thus

\[ y = x - z = \frac{p}{q} - \frac{s}{t} = \frac{pt - sq}{qt}. \]

Since \( p, q, s, t \in \mathbb{Z}, \)

\( pt - sq \in \mathbb{Z} \) and \( qt \in \mathbb{Z}. \) Since

\( q \neq 0 \) and \( t \neq 0, \)

\( qt \neq 0. \) Thus \( y \in \mathbb{Q}. \) This is a contradiction. \( \therefore \) \( z \notin \mathbb{R} \setminus \mathbb{Q}. \)

**Uniqueness**

Assume that there exists \( z_1, z_2 \in \mathbb{R} \setminus \mathbb{Q} \) so that

\[ y + z_1 = x \quad (1) \]

\[ y + z_2 = x \quad (2) \]

Subtracting \( (2) \) from \( (1) \) gives that \( z_1 - z_2 = 0. \) So \( z_1 = z_2. \)
(V (a, b, c) ∈ \mathbb{N}^3) (\text{c} | \text{a} \land \text{c} | \text{b} \Rightarrow \text{c} | \text{GCD}(a, b))

\text{Fix } a, b, c \in \mathbb{N}. \text{ Let } c | a \text{ and } c | b. \text{ Thus }
\begin{align}
ck &= a \\
cl &= b
\end{align}
\tag{1}
\text{for some } k, l \in \mathbb{N}.

\text{Let } d = \text{GCD}(a, b). \text{ By the division algorithm }
\begin{align}
d &= ax + by \tag{2}
\end{align}
\text{for some } x, y \in \mathbb{Z}. \text{ Substituting (1) into (2) gives that }
\begin{align}
d &= ax + by = (ck)x + (cl)y = c(kx + ly) = c j
\end{align}
\text{where } j = kx + ly. \text{ Since } k, l \in \mathbb{N} \text{ and } x, y \in \mathbb{Z},
\text{we know that } j \in \mathbb{Z}. \text{ Thus } c | d. \quad \square