NAME:  
Prof. Girardi  04.11.06  Exam 2  Math 300  

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>POINTS</th>
<th>Problem Inspiration</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11</td>
<td>Ch 1 handout pages 8 and 9</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>Ch 1 handout page 12</td>
</tr>
</tbody>
</table>
| 3 - 1st choice | 9 | 3a: hand-in HW § 1.5 # 3g  
|           |        | 3b: look-at HW § 1.6 # 7k  
|           |        | 3c: in-class HW § 1.7 # 3d  
|           |        | 3d: in-class HW § 1.7 # 11a |
| 3 - 2nd choice | 9 | |
| 3 - 3rd choice | 9 | |
| TOTAL    | 50     | |

INSTRUCTIONS:  
(1) The MARK BOX indicates the problems along with their points.  
   Check that your copy of the exam has all of the problems.  
(2) When applicable put your answer on/in the line/box provided.  
   Show your work UNDER the provided line/box.  
   If no such line/box is provided, then box your answer.  
(3) For problems that request a proof, write a neat FORMAL proof and use only logic and the  
    definitions of the concepts involved (i.e., do not quote a problem from the book). You may  
    include your skeleton, if you so wish.  
(4) During this exam, do not leave your seat. If you have a question, raise your hand. When  
    you finish: turn your exam over, put your pencil down, and raise your hand.  
(5) You may not use a calculator, books, personal notes.  
(6) This exam covers (from A Transition to Advanced Mathematics by Smith, Eggen, and St.  
    Andre: 6th ed.): Sections 1.1 – 1.7.  

Numbers:  
- real numbers = \( \mathbb{R} \)  
- positive real numbers = \( \mathbb{R}^+ = \{ x \in \mathbb{R}: x > 0 \} \)  
- natural numbers = \( \mathbb{N} = \{ 1, 2, 3, 4, \ldots \} \)  
- integers = \( \mathbb{Z} = \{ 0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots \} \)  
- rational numbers = \( \mathbb{Q} = \{ \frac{p}{q} \in \mathbb{R}: p, q \in \mathbb{Z} \text{ and } q \neq 0 \} \)  

Throughout this exam:  
(1) \( P, Q, \) and \( R \) be propositions.  
(2) \( P(x_1, \ldots, x_n), Q(x_1, \ldots, x_n), \) and \( R(x_1, \ldots, x_n) \) are open sentences with variables \( x_1, \ldots, x_n \).
1. Make true statements by filling in the blanks with the appropriate number between 1 and 11 from below. Use each number once, and only once.

   (1) \( (P \implies Q) \land (P \implies R) \)
   (2) \( (P \implies Q) \land (Q \implies P) \)
   (3) \( (P \implies R) \land (Q \implies R) \)
   (4) \( (P \land \sim Q) \implies R \)
   (5) \( P \implies \sim Q \)
   (6) \( \sim Q \implies \sim P \)
   (7) \( (\sim P) \land (\sim Q) \)
   (8) \( (P \land \sim Q) \)
   (9) \( (\forall x)[\sim P(x)] \)
   (10) \( (\exists x)[\sim P(x)] \)
   (11) \( (\exists x)[P(x) \land (\forall y)(P(y) \implies x = y)] \)

\[
\begin{align*}
P \implies Q & \text{ is equivalent to } \quad \underline{} \\
P \iff Q & \text{ is equivalent to } \quad \underline{} \\
\sim (P \implies Q) & \text{ is equivalent to } \quad \underline{} \\
\sim (P \land Q) & \text{ is equivalent to } \quad \underline{} \\
\sim (P \lor Q) & \text{ is equivalent to } \quad \underline{} \\
(\exists x)P(x) & \text{ is equivalent to } \quad \underline{} \\
(\forall x)P(x) & \text{ is equivalent to } \quad \underline{} \\
(\forall x)P(x) & \text{ is equivalent to } \quad \underline{} \\
(\exists x)P(x) & \text{ is equivalent to } \quad \underline{} \\
\end{align*}
\]
2. Circle T if and only if the statement is TRUE. Circle F if and only if the statement is FALSE.

\begin{align*}
T & \quad F \quad (\forall x)(\forall y)P(x, y) \implies (\forall y)(\forall x)P(x, y) \\
T & \quad F \quad (\exists x)(\exists y)P(x, y) \implies (\exists y)(\exists x)P(x, y) \\
T & \quad F \quad (\forall x)P(x) \implies (\exists x)P(x) \\
T & \quad F \quad (\exists x)P(x) \implies (\forall x)P(x) \\
T & \quad F \quad (\forall x)[P(x) \lor Q(x)] \implies [(\forall x)P(x) \lor (\forall x)Q(x)] \\
T & \quad F \quad [(\forall x)P(x) \lor (\forall x)Q(x)] \implies (\forall x)[P(x) \lor Q(x)] \\
T & \quad F \quad (\forall x)[P(x) \implies Q(x)] \implies [(\forall x)P(x) \implies (\forall x)Q(x)] \\
T & \quad F \quad [(\forall x)P(x) \implies (\forall x)Q(x)] \implies (\forall x)[P(x) \implies Q(x)] \\
T & \quad F \quad (\forall y)(\exists x)P(x, y) \implies (\exists x)(\forall y)P(x, y) \\
T & \quad F \quad (\exists x)(\forall y)P(x, y) \implies (\forall y)(\exists x)P(x, y) \\
T & \quad F \quad (\forall x)[P(x) \land Q(x)] \implies [(\forall x)P(x) \land (\forall x)Q(x)] \\
T & \quad F \quad [(\forall x)P(x) \land (\forall x)Q(x)] \implies (\forall x)[P(x) \land Q(x)]
\end{align*}
3. Do THREE problems from problems: 3a, 3b, 3c, and 3d. Fill-in the blanks on this sheet of paper but include your proof parts on a separate sheet of paper (with your name on each sheet please).

Circle THREE: I am doing problems 3a, 3b, 3c, 3d

3a. Let \( x \in \mathbb{Z} \). If \( 8 \) does not divide \( x^2 - 1 \), then \( x \) is even.

- Let \( a, b, c \in \mathbb{Z} \). By definition:

  \( a \) divides \( b \), denoted by \( a \mid b \), if and only if \( \frac{b}{a} \) is an integer.

  \( c \) is even if and only if \( c \) is divisible by \( 2 \).

  \( c \) is odd if and only if \( c \) is not divisible by \( 2 \).

- Symbolically write, using quantifiers, statement 3a.

- Prove statement 3a.

3b. There exists an odd integer \( M \) such that for each real number \( r \) larger than \( M \), \( \frac{1}{2r} < 0.001 \).

- By definition, an integer \( M \) is odd if and only if \( M \) is not divisible by \( 2 \).

- Symbolically write, using quantifiers, statement 3b.

- Prove statement 3b.
3c. For each rational number \(x\) and each irrational number \(y\), there is a unique irrational number \(z\) such that \(y + z = x\).

- Let \(a\) and \(b\) be real numbers. By definition:

\[
a \text{ is rational if and only if } \quad \text{______________________________}
\]

\[
b \text{ is irrational if and only if } \quad \text{______________________________}.
\]

- Symbolically write, using quantifiers, statement 3c.

\[
\quad \text{______________________________}.
\]

- Prove statement 3c. You must carefully prove, using only the definitions above (and not using another homework problem), that your \(z\) is indeed irrational.

3d. Let \(a, b, c, d \in \mathbb{N}\) and \(d = GCD(a, b)\). If \(c\) divides \(a\) and \(c\) divides \(b\), then \(c\) divides \(d\).

- By definition:

\[
c \text{ divides } d, \text{ denoted } c|d, \text{ if and only if } \quad \text{______________________________}
\]

\[
d \text{ is the greatest common divisor of } a \text{ and } b, \text{ denoted by } d = GCD(a, b), \text{ if and only if } \\
\quad \text{______________________________}.
\]

- Symbolically write, using quantifiers, statement 3d.

\[
\quad \text{______________________________}.
\]

- Prove statement 3d.