These problems are samples of final-like problems.

These problems do not constitute a comprehensive review for the final.

1. Let the curve C be the helix parameterized by $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \le t \le 2\pi$ and let $f(x,y,z) = x^2 + y^2 + z^2$. Evaluate the line integral $\int_C f(x,y,z)\, ds$.

See Thomas 15th, §16.1.

$\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$. $\quad f(x,y,z) = x^2 + y^2 + z^2 = \cos^2 t + \sin^2 t + t^2 = 1 + t^2$.

$$\int_C f(x,y,z)\, ds = \int_0^{2\pi} f(\vec{r}(t))\, \|\vec{r}\,'(t)\|\, dt = \int_0^{2\pi} (1+t^2)\sqrt{2}\, dt = \sqrt{2}\left(t + \tfrac{t^3}{3}\,\Big|_{t=0}^{t=2\pi}\right) = \boxed{\frac{2\sqrt{2}\,\pi}{3}\left(3+4\pi^2\right)}$$

2. Evaluate the line integral $\int_C x^2 y\, dx + (x-2y)\, dy$ where C is the part of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.

See Thomas 15th, §16.2. $\quad \vec{r}(t) = \langle t, t^2 \rangle = \langle x(t), y(t) \rangle$ where $0 \le t \le 1$.

So $x(t) = t$ and $y(t) = t^2$. And $dx = dt$ and $dy = 2t\, dt$. So $\int_C x^2 y\, dx + (x-2y)\, dy =$

$$\int_0^1 t^2\, t^2\, dt + (t - 2t^2)\, 2t dt = \int_0^1 (\, t^4 + 2t^2 - 4t^3\,)\, dt = \left(\tfrac{t^5}{5} + \tfrac{2t^3}{3} - t^4\right)\Big|_0^1 = \boxed{-\frac{2}{15}}$$

3. Evaluate the line integral $\int_C \left(2y + \sqrt{1+x^5}\right) dx + \left(5x - e^{y^2}\right) dy$ where C is the circle $x^2 + y^2 = 4$ oriented counterclockwise.

See Thomas 15th, §16.4 Green's Theorem. Let $P(x,y) = 2y + \sqrt{1+x^5}$ and $Q(x,y) = 5x - e^{y^2}$. So $\frac{\partial Q}{\partial x}(x,y) = 5$ and $\frac{\partial P}{\partial y}(x,y) = 2$. Let the region $R = \{(x,y) : x^2 + y^2 \le 4\}$. So $\int_C \left(2y + \sqrt{1+x^5}\right) dx + \left(5x - e^{y^2}\right) dy \overset{\text{Green's}}{=} \iint_R (5-2)\, dA = 3\iint_R dA = 3\,(\text{area of circle with } r = 2) = 3\,(4\pi) = \boxed{12\pi}$

4. Find the equation of the plane that contains the points $(0,1,2)$, $(-1,2,3)$, and $(-4,-1,2)$. Demonstrate (i.e., check) that your answer is correct.

Let $P = (0,1,2)$, $Q = (-1,2,3)$, and $R = (-4,-1,2)$. We see that $\overrightarrow{PQ} = -\vec{\imath} + \vec{\jmath} + \vec{k}$ and $\overrightarrow{PR} = -4\vec{\imath} - 2\vec{\jmath}$. Thus,

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \det \begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -1 & 1 & 1 \\ -4 & -2 & 0 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \vec{\imath} - \det \begin{bmatrix} -1 & 1 \\ -4 & 0 \end{bmatrix} \vec{\jmath} + \det \begin{bmatrix} -1 & 1 \\ -4 & -2 \end{bmatrix} \vec{k}$$

$$= 2\vec{\imath} - 4\vec{\jmath} + 6\vec{k}.$$

The plane through $(0,1,2)$ perpendicular to $2\vec{\imath} - 4\vec{\jmath} + 6\vec{k}$ is

$$2(x-0) - 4(y-1) + 6(z-2) = 0.$$

Our answer is the same as

$$x - 2(y-1) + 3(z-2) = 0$$

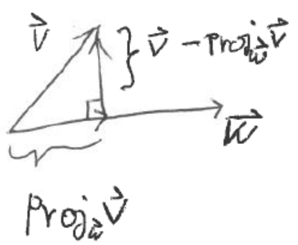
or

$$\boxed{x - 2y + 3z = 4.}$$

Check

$$\begin{aligned} 0 - 2(1) + 3(2) &= 4 \\ -1 - 2(2) + 3(3) &= 4 \\ -4 - 2(-1) + 3(2) &= 4 \end{aligned}$$

5. Express $\vec{v} = 3\vec{\imath} + 5\vec{\jmath} + \vec{k}$ as the sum of a vector parallel to $\vec{w} = \vec{\imath} + 2\vec{\jmath} - \vec{k}$ and a vector perpendicular to $\vec{w}$. Demonstrate (i.e., check) that your answer is correct.



We calculate

$$\begin{aligned} \operatorname{proj}_{\vec{w}} \vec{v} &= \left(\vec{v} \cdot \frac{\vec{w}}{\|\vec{w}\|}\right) \frac{\vec{w}}{\|\vec{w}\|} = \left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\right) \vec{w} = \frac{3 + 10 - 1}{1 + 4 + 1}(\vec{\imath} + 2\vec{\jmath} - \vec{k}) \\ &= \frac{12}{6}(\vec{\imath} + 2\vec{\jmath} - \vec{k}) = 2\vec{\imath} + 4\vec{\jmath} - 2\vec{k}. \end{aligned}$$

We see that

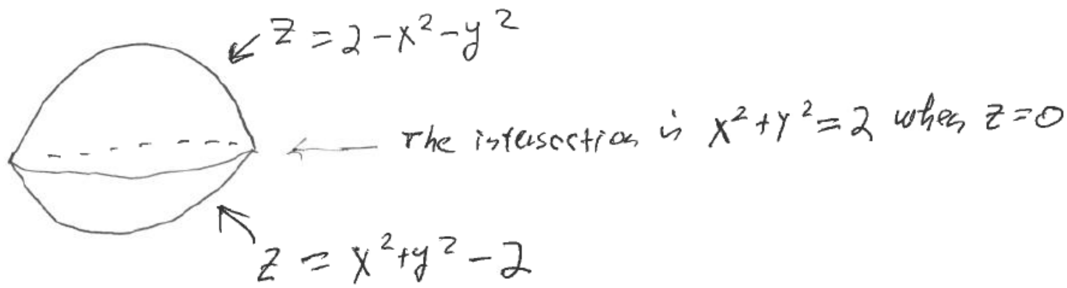
$$\vec{v} - \operatorname{proj}_{\vec{w}} \vec{v} = 3\vec{\imath} + 5\vec{\jmath} + \vec{k} - (2\vec{\imath} + 4\vec{\jmath} - 2\vec{k}) = \vec{\imath} + \vec{\jmath} + 3\vec{k}.$$

We conclude that

$$\boxed{\begin{array}{l} \vec{v} = (2\vec{\imath} + 4\vec{\jmath} - 2\vec{k}) + (\vec{\imath} + \vec{\jmath} + 3\vec{k}) \\ \text{with } 2\vec{\imath} + 4\vec{\jmath} - 2\vec{k} \text{ parallel to } \vec{w} \\ \text{and } \vec{\imath} + \vec{\jmath} + 3\vec{k} \text{ perpendicular to } \vec{w}. \end{array}}$$

Check. It is clear that $(2\vec{\imath}+4\vec{\jmath}-2\vec{k})+(\vec{\imath}+\vec{\jmath}+3\vec{k}) = 3\vec{\imath}+5\vec{\jmath}+1\vec{k}$. It is clear that $2\vec{\imath}+4\vec{\jmath}-2\vec{k}$ is parallel to $\vec{\imath} + 2\vec{\jmath} - \vec{k}$. We compute $(\vec{\imath} + \vec{\jmath} + 3\vec{k}) \cdot (\vec{\imath} + 2\vec{\jmath} - \vec{k}) = 1 + 2 - 3 = 0$.✓

6. Find the volume between $z = 2 - x^2 - y^2$ and $z = x^2 + y^2 - 2$. (Draw a meaningful picture.)



The intersection is the circle $x^2 + y^2 = 2$ in the xy-plane.

$$\text{Volume} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sqrt{2}} ((2-r^2) - (r^2-2)) r \, dr \, d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sqrt{2}} \left(4r - 2r^3\right) \, dr \, d\theta$$
$$= \int_{\theta=0}^{\theta=2\pi} \left(2r^2 - \frac{r^4}{2}\right) \Big|_{r=0}^{r=\sqrt{2}} d\theta = 2\pi(4-2) = \boxed{4\pi}.$$

7. Compute $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2} \, dy \, dx$.

We do the problem in polar coordinates. We are integrating over the quarter of the unit circle which is in the first quadrant.

$$\int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=1} e^{r^2} \, r \, dr \, d\theta = \int_{\theta=0}^{\theta=\pi/2} \left[\frac{1}{2} e^{r^2} \Big|_{r=0}^{r=1} \right] d\theta = \int_{\theta=0}^{\theta=\pi/2} \frac{1}{2} (e-1) \, d\theta = \frac{\pi}{2} \frac{1}{2} (e-1) = \boxed{\frac{\pi}{4}(e-1)}.$$

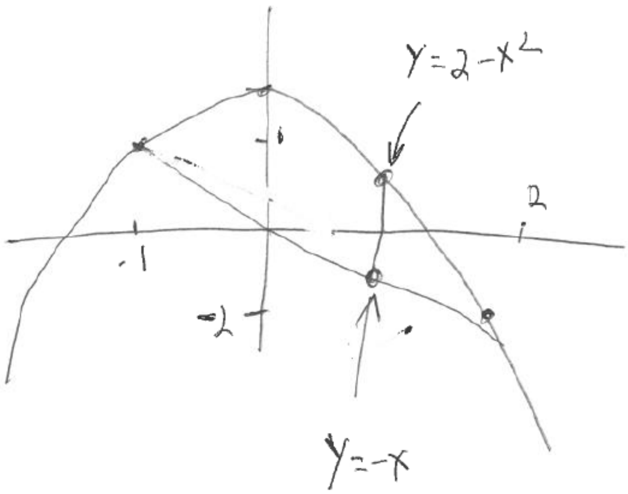
8. Find the directional derivative of $f(x,y,z) = x^3 - xy^2 - z$ at the point $P = (1,1,0)$, in the direction of $\vec{v} = 2\vec{\imath} - 3\vec{\jmath} + 6\vec{k}$.

$$(D_{\vec{v}} f)|_P = (\vec{\nabla} f)|_P \cdot \frac{\vec{v}}{\|\vec{v}\|} = ((3x^2 - y^2)\vec{\imath} - 2xy\vec{\jmath} - \vec{k}) \,|_{(1,1,0)} \cdot \frac{2\vec{\imath} - 3\vec{\jmath} + 6\vec{k}}{\sqrt{4+9+36}}$$
$$= (2\vec{\imath} - 2\vec{\jmath} - \vec{k}) \cdot \frac{2\vec{\imath} - 3\vec{\jmath} + 6\vec{k}}{7} = \frac{4+6-6}{7} = \boxed{\frac{4}{7}}.$$

9. Find all points (x,y) where a local maximum, local minimum, or saddle point occurs for the function $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$.

We compute $f_x(x,y) = y - 2x - 2$ and $f_y(x,y) = x - 2y - 2$. Both partial derivatives are zero when $y = 2x + 2$ and $0 = x - 2(2x+2) - 2$. So $3x = -6$, $x = -2$, and $y = -2$. The point $(-2,-2)$ is the only critical point. We apply the second derivative test. We see that $f_{xx}(x,y) = -2$, $f_{xy}(x,y) = 1$, and $f_{yy}(x,y) = -2$. Thus $D = f_{xx} f_{yy} - f_{xy}^2 = 4 - 1 = 3 > 0$ and $f_{xx} < 0$. We conclude that $\boxed{\text{a local maximum occurs when } (x,y) = (-2,-2).}$

10. Find the area of the region bounded by $y + x^2 = 2$ and $y + x = 0$. (Draw a meaningful picture.)



Observe that $y = 2 - x^2$ is a parabola with with vertex at $(0, 2)$ opening downward and $y = -x$ is the line through the origin with slope -1. These two curves intersect at $(2, -2)$ and $(-1, 1)$. For each fixed x, with $-1 \leq x \leq 2$, y goes from $-x$ to $2 - x^2$. The area is

$$\int_{x=-1}^{x=2} \int_{y-x}^{y=2-x^2} dy\, dx = \int_{x=-1}^{x=2} (2 - x^2 + x)\, dx = (2x - \frac{x^3}{3} + \frac{x^2}{2})\Big|_{x=-1}^{x=2} = 4 - \frac{8}{3} + 2 - (-2 + \frac{1}{3} + \frac{1}{2}) = \boxed{\frac{9}{2}}.$$