These problems are samples of final-like problems.
These problems do not constitute a comprehensive review for the final.

1. Find the equation of the plane that contains the points $(0, 1, 2)$, $(-1, 2, 3)$, and $(-4, -1, 2)$. Demonstrate (i.e., check) that your answer is correct.

Soln. Let $P = (0, 1, 2)$, $Q = (-1, 2, 3)$, and $R = (-4, -1, 2)$. We see that $\overrightarrow{PQ} = -\vec{\imath} + \vec{\jmath} + \vec{k}$ and $\overrightarrow{PR} = -4\vec{\imath} - 2\vec{\jmath}$. Thus,

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \det \begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -1 & 1 & 1 \\ -4 & -2 & 0 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \vec{\imath} - \det \begin{bmatrix} -1 & 1 \\ -4 & 0 \end{bmatrix} \vec{\jmath} + \det \begin{bmatrix} -1 & 1 \\ -4 & -2 \end{bmatrix} \vec{k}$$

$$= 2\vec{\imath} - 4\vec{\jmath} + 6\vec{k}.$$

The plane through $(0, 1, 2)$ perpendicular to $2\vec{\imath} - 4\vec{\jmath} + 6\vec{k}$ is

$$2(x - 0) - 4(y - 1) + 6(z - 2) = 0.$$

Our answer is the same as

$$x - 2(y - 1) + 3(z - 2) = 0$$

or

$$\boxed{x - 2y + 3z = 4.}$$

Check

$$\begin{aligned} 0 - 2(1) + 3(2) &= 4 \\ -1 - 2(2) + 3(3) &= 4 \\ -4 - 2(-1) + 3(2) &= 4 \end{aligned}$$

2. Find the cosine of the angle θ between the vectors $\vec{A} = \langle 1, 2, 2 \rangle$ and $\vec{B} = \langle -3, 4, 0 \rangle$.

Soln.

The *dot product* $\mathbf{A} \cdot \mathbf{B}$ of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined to be the product of their lengths and the cosine of the angle between them. This definition can be written as

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta, \tag{1}$$

If $\mathbf{A}$ and $\mathbf{B}$ are nonzero vectors, the definition (1) can be written in the form

$$\cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}. \tag{9}$$

Example 2 Find the cosine of the angle θ between the vectors $\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{B} = -3\mathbf{i} + 4\mathbf{j}$.

Solution It is clear that

$$|\mathbf{A}| = \sqrt{1 + 4 + 4} = 3, \qquad |\mathbf{B}| = \sqrt{9 + 16} = 5, \qquad \mathbf{A} \cdot \mathbf{B} = -3 + 8 + 0 = 5.$$

Therefore by (9) we have

$$\cos \theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{5}{3 \cdot 5} = \frac{1}{3}.$$

If we want the angle θ itself, we can use a calculator to find that $\theta \cong 70.5^\circ$.

3. Let c be a constant (real number). Consider the two vectors $\vec{A} = \langle 1, -2, 2 \rangle$ and $\vec{B} = \langle -1, 0, c \rangle$. Find a value of c so that $\vec{A} \perp \vec{B}$.
Soln.

Example 3 Compute the cosine of the angle θ between $\mathbf{A}$ and $\mathbf{B}$ if $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{B} = -\mathbf{i} + c\mathbf{k}$, and find a value of c for which $\mathbf{A} \perp \mathbf{B}$.

Solution We have

$$|\mathbf{A}| = \sqrt{1+4+4} = 3, \qquad |\mathbf{B}| = \sqrt{1+c^2}, \qquad \mathbf{A}\cdot\mathbf{B} = -1 + 2c,$$

so

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{2c-1}{3\sqrt{1+c^2}}.$$

When $c = \frac{1}{2}$, this quantity has the value 0, and hence the vectors are perpendicular.

Another Soln. $\vec{A} \perp \vec{B} \Leftrightarrow \vec{A}\cdot\vec{B} = 0$. Here $\vec{A}\cdot\vec{B} = -1 + 2c$. So want $-1 + 2c$. So take $c = \frac{1}{2}$.

4. Let $\vec{u} = \langle -1, 5 \rangle$ and $\vec{v} = \langle 3, 3 \rangle$. Find the vector projection of $\vec{u}$ onto $\vec{v}$.
Soln. See solution to next question.

5. Let $\vec{u} = \langle -1, 5 \rangle$ and $\vec{v} = \langle 3, 3 \rangle$. Find the (scalar) component of $\vec{u}$ onto $\vec{v}$.
Soln. Helpful: 12.3 Vector Projections and Scalar Components Summary handout, which has the following about Vector Projection and Scalar Component of a vector $\vec{A}$ onto a nonzero vector $\vec{B}$.
The vector projection of $\vec{A}$ onto $\vec{B}$ is

$$\overrightarrow{\text{proj}}_{\vec{B}}\,\vec{A} \stackrel{\text{def}}{=} \left(\vec{A}\cdot\frac{\vec{B}}{\|\vec{B}\|}\right)\frac{\vec{B}}{\|\vec{B}\|} = \left(\frac{\vec{A}\cdot\vec{B}}{\|\vec{B}\|^2}\right)\vec{B} = \left(\frac{\vec{A}\cdot\vec{B}}{\vec{B}\cdot\vec{B}}\right)\vec{B}\,.$$

The scalar component of $\vec{A}$ in the direction of $\vec{B}$ is

$$\text{comp}_{\vec{B}}\,\vec{A} \stackrel{\text{def}}{=} \vec{A}\cdot\frac{\vec{B}}{\|\vec{B}\|} = \frac{\vec{A}\cdot\vec{B}}{\|\vec{B}\|} \underset{\text{denoted}}{\stackrel{\text{also}}{=}} \text{scal}_{\vec{B}}\,\vec{A}\,.$$

Soln to 4. $\overrightarrow{\text{proj}}_{\vec{v}}\,\vec{u} = \overrightarrow{\text{proj}}_{\langle 3,3\rangle}\,\langle -1, 5 \rangle = \left(\frac{\langle -1,5\rangle\cdot\langle 3,3\rangle}{\|\langle 3,3\rangle\|^2}\right)\langle 3,3\rangle = \left(\frac{-3+15}{3^2+3^2}\right)\langle 3,3\rangle = \frac{2}{3}\langle 3,3\rangle = \langle 2,2\rangle$.

Soln to 5. $\overrightarrow{\text{comp}}_{\vec{v}}\,\vec{u} = \overrightarrow{\text{comp}}_{\langle 3,3\rangle}\,\langle -1,5\rangle = \frac{\langle -1,5\rangle\cdot\langle 3,3\rangle}{\|\langle 3,3\rangle\|} = \frac{-3+15}{\sqrt{3^2+3^2}} = \frac{12}{3\sqrt{2}} = \frac{4}{\sqrt{2}} \underset{\text{ok}}{\stackrel{\text{also}}{=}} 2\sqrt{2}.$

6. Find an equation of the plane tangent to the surface defined by $3xy + z^2 = 4$ at the point $(1,1,1)$.
Soln. Helpful: 14.6 Tangent Planes handout.

Compute the equation of the plane tangent to the surface defined by $3xy + z^2 = 4$ at $(1, 1, 1)$.

Here $f(x, y, z) = 3xy + z^2$ and $\nabla f = (3y, 3x, 2z)$, which at $(1, 1, 1)$ is the vector $(3, 3, 2)$. Thus, the tangent plane is

$$(3, 3, 2) \cdot (x - 1, y - 1, z - 1) = 0;$$

that is,

$$3x + 3y + 2z = 8. \quad \blacktriangle$$

7. Find the arc length of the curve $\vec{c}\,(t) = \langle t - \sin t, 1 - \cos t \rangle$ for $0 \le t \le 2\pi$.
Integration Hint: the Half-Angle Formulas are $\cos^2 x = \frac{1+\cos(2x)}{2}$ and $\sin^2 x = \frac{1-\cos(2x)}{2}$.
Soln. Helpful: 13.3 Summary of Arc Length handout.

Consider the point with position function

$$\mathbf{c}(t) = (t - \sin t, 1 - \cos t),$$

which traces out the cycloid discussed in Section 2.4 (see Figure 2.4.6). Find the velocity, the speed, and the length of one arch.

The velocity vector is $\mathbf{c}'(t) = (1 - \cos t, \sin t)$, so the speed of the point $\mathbf{c}(t)$ is

$$\|\mathbf{c}'(t)\| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}.$$

Hence, $\mathbf{c}(t)$ moves at variable speed although the circle rolls at constant speed. Furthermore, the speed of $\mathbf{c}(t)$ is zero when t is an integral multiple of 2π. At these values of t, the y coordinate of the point $\mathbf{c}(t)$ is zero, and so the point lies on the x axis. The arc length of one cycle is

$$\begin{aligned}
L(\mathbf{c}) &= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = 2\int_0^{2\pi} \sqrt{\frac{1 - \cos t}{2}}\, dt \\
&= 2\int_0^{2\pi} \sin\frac{t}{2}\, dt \left(\text{because } 1 - \cos t = 2\sin^2\frac{t}{2} \text{ and } \sin\frac{t}{2} \ge 0 \text{ on } [0, 2\pi]\right) \\
&= 4\left(-\cos\frac{t}{2}\right)\Bigg|_0^{2\pi} = 8. \quad \blacktriangle
\end{aligned}$$

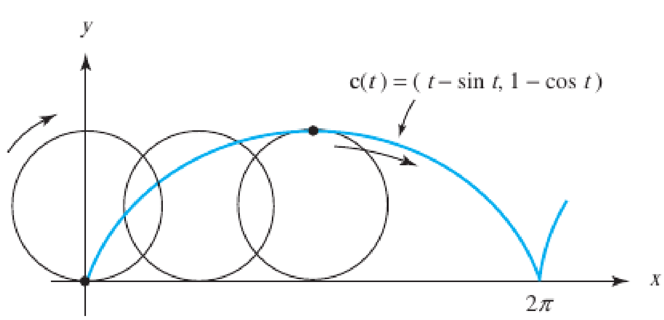


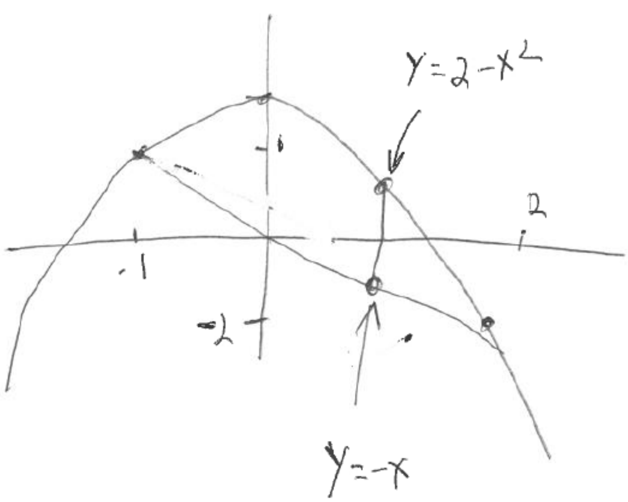
figure **2.4.6** The curve traced by a point moving on the rim of a rolling circle is called a cycloid.

8. Find the directional derivative of $f(x,y,z) = x^3 - xy^2 - z$ at the point $P = (1,1,0)$, in the direction of $\vec{v} = 2\vec{\imath} - 3\vec{\jmath} + 6\vec{k}$.
Soln.

$$(D_{\vec{v}}f)|_P = (\vec{\nabla} f)|_P \cdot \frac{\vec{v}}{\|\vec{v}\|} = ((3x^2 - y^2)\vec{\imath} - 2xy\vec{\jmath} - \vec{k})\,|_{(1,1,0)} \cdot \frac{2\vec{\imath} - 3\vec{\jmath} + 6\vec{k}}{\sqrt{4+9+36}}$$
$$= (2\vec{\imath} - 2\vec{\jmath} - \vec{k}) \cdot \frac{2\vec{\imath} - 3\vec{\jmath} + 6\vec{k}}{7} = \frac{4+6-6}{7} = \boxed{\frac{4}{7}}.$$

9. Find all points (x,y) where a local maximum, local minimum, or saddle point occurs for the function $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$.
Soln. We compute $f_x(x,y) = y - 2x - 2$ and $f_y(x,y) = x - 2y - 2$. Both partial derivatives are zero when $y = 2x + 2$ and $0 = x - 2(2x+2) - 2$. So $3x = -6$, $x = -2$, and $y = -2$. The point $(-2,-2)$ is the only critical point. We apply the second derivative test. We see that $f_{xx}(x,y) = -2$, $f_{xy}(x,y) = 1$, and $f_{yy}(x,y) = -2$. Thus $D = f_{xx}f_{yy} - f_{xy}^2 = 4 - 1 = 3 > 0$ and $f_{xx} < 0$. We conclude that $\boxed{\text{a local maximum occurs when } (x,y) = (-2,-2).}$

10. Find the area of the region bounded by $y + x^2 = 2$ and $y + x = 0$. (Draw a meaningful picture.)
Soln.



Observe that $y = 2 - x^2$ is a parabola with with vertex at $(0,2)$ opening downward and $y = -x$ is the line through the origin with slope -1. These two curves intersect at $(2,-2)$ and $(-1,1)$. For each fixed x, with $-1 \le x \le 2$, y goes from $-x$ to $2 - x^2$. The area is

$$\int_{x=-1}^{x=2}\int_{y-x}^{y=2-x^2} dy\,dx = \int_{x=-1}^{x=2} (2 - x^2 + x)\,dx = (2x - \frac{x^3}{3} + \frac{x^2}{2})\Big|_{x=-1}^{x=2} = 4 - \frac{8}{3} + 2 - (-2 + \frac{1}{3} + \frac{1}{2}) = \boxed{\frac{9}{2}}.$$