Appendix B

B.1 Determinants

A **matrix** is a rectangular array of numbers. For example,

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix}$$

is a matrix with two rows and three columns. We call A a "2 by 3" matrix. More generally, an "m by n matrix" is one that has m rows and n columns.

The element in the ith row and jth column of a matrix is represented by a_{ij}. In the example above, we have

$$\begin{aligned} a_{11} &= 2, & a_{12} &= 1, & a_{13} &= 3, \\ a_{21} &= 1, & a_{22} &= 0, & a_{23} &= -2. \end{aligned}$$

If A is an n by n matrix, then we associate A with a number called the **determinant** of A, written sometimes as $\det A$ and sometimes as $|A|$ with vertical bars (which do not mean absolute value). For $n = 1$ and $n = 2$ we have these definitions:

$$\det[a] = a, \qquad \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc. \quad \frac{\text{other}}{\text{notation}} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \tag{1}$$

For a 3 by 3 matrix, we define (note the one subtraction)

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{2}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

In Equation (2) there are some determinants of 2 by 2 matrices – each of those matrices is obtained by deleting one row and one column of the original 3 by 3 matrix.

Remark Notice the big difference between "the brackets":

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \leftarrow \text{brackets are: } [\cdots]$$

while

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \leftarrow \text{brackets are: } |\cdots|$$

$\boxed{\text{So use } [\cdot] \text{ for a matrix and } |\cdot| \text{ for determinate of a matrix.}}$

$\boxed{\text{like in absolute value}}$

EXAMPLE 1 Find det (A) when

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 0 \end{bmatrix},$$

Recall notation .

Solution Helpful. Think

$+$ $\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 0 \end{bmatrix}$ $\qquad -$ $\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 0 \end{bmatrix}$ $\qquad +$ $\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 0 \end{bmatrix}$

Recall is determinate

recall formula... "middle" one is subtracted

$$\det A = (^{+}2)\begin{vmatrix} -1 & -2 \\ 3 & 0 \end{vmatrix} - (^{+}1)\begin{vmatrix} 3 & -2 \\ 2 & 0 \end{vmatrix} + (^{+}3)\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}$$

now compute the determines of the three 2x2 matrices,

Recall $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

$$= (2)\big((-1)(0) - (-2)(3)\big) - (1)\big((3)(0) - (-2)(2)\big) + 3\big((3)(3) - (-1)(2)\big)$$

$$= (2)(6) - (1)(4) + 3(11)$$

$$= 12 - 4 + 33$$

$$= \boxed{41}$$