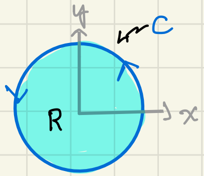
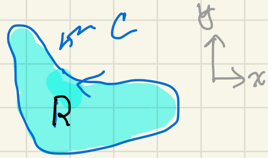
§ 16.4 Green's Theorem

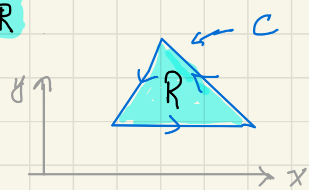
Sample (planar) curves C and regions R



Sample 1



Sample 2



Sample 3

Green's Theorem

Let the curve C in the plane be
positively-oriented, piecewise-smooth, simple, closed.
Let R be the region bounded by C.
Let the (scalar-valued) functions P and Q have continuous 1st partial derivative on an open set that contains D.
Then

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}. \quad \text{(Green)}$$

Remark:
Letting $\vec{F} \overset{\text{def}}{=} \langle P, Q \rangle$ and $\vec{r} : [a,b] \to \mathbb{R}^2$ trace out C, in §16.2 we got

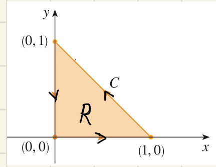
$$\underbrace{\oint_C P\,dx + Q\,dy}_{\text{to calculate this without (Green), would need to find } \vec{r}(t).} = \int_C \vec{F} \cdot d\vec{r} = \int_{t=a}^{t=b} \left[\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \right] dt. \quad (2)$$

Def in Green's Thm for the curve C traced out by $\vec{r}$.

1. positively oriented: if walk along C then R is to your LEFT (TL: unit circle)
2. piecewise-smooth: except for a finite number of pts, $\vec{r}$ is smooth ($\vec{r}\,'$ cont. & $\neq \vec{0}$)
3. simple: C does not intersect itself, i.e. if $a < t_1 < t_2 < b$ then $\vec{r}(t_1) \neq \vec{r}(t_2)$
4. closed: $\vec{r}(a) = \vec{r}(b)$

Recall If $-C$ denotes "C traced out in the opposite direction", then $\int_{-C} = -\int_C$

Ex 1 For the triangular curve C consisting of the line segments from $(0,0)$ to $(1,0)$, followed by from $(1,0)$ to $(0,1)$, then from $(0,1)$ to $(0,0)$, evaluate the line integral $\int_C x^4\,dx + xy\,dy$.



Soln

$$\int_C \underbrace{x^4}_{P}\,dx + \underbrace{xy}_{Q}\,dy =$$

Ex 2 For the C which is the circle $x^2+y^2=9$ traced counterclockwise, evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$.

Ex 3 Do Ex 2 but with C traced clockwise. Soln.

Using Green's Theorem to find Area

16.4.3

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}.$$ (Green)

If $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 1$, then

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R dA = \text{area of } R}$$

The $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 1$ in each of the following situations.

1. $\langle P(x,y), Q(x,y) \rangle = \langle 0, x \rangle$
2. $\langle P(x,y), Q(x,y) \rangle = \langle -y, 0 \rangle$
3. $\langle P(x,y), Q(x,y) \rangle = \langle -\frac{1}{2}y, \frac{1}{2}x \rangle$

So (when C is parametrized with positive orientation)

$$\text{area of } R = \oint_C x\,dy = \oint_C -y\,dx = \frac{1}{2}\oint x\,dy - y\,dx$$

Ex 4 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Soln