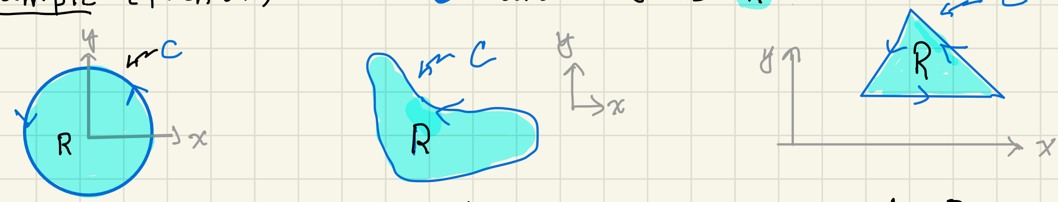
§ 16.4 Green's Theorem

Sample (planar) curves C and regions R



Sample 1 Sample 2 Sample 3

Green's Theorem

Let the curve C in the plane be positively-oriented, piecewise-smooth, simple, closed.
Let R be the region bounded by C.
Let the (scalar-valued) functions P and Q have continuous 1st partial derivative on an open set that contains D.

Then

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA} . \qquad \text{(Green)}$$

Remark:
Letting $\vec{F} \overset{\text{def}}{=} \langle P, Q \rangle$ and $\vec{r} : [a,b] \to \mathbb{R}^2$ trace out C, in §16.2 we got

$$\underbrace{\oint_C P\,dx + Q\,dy}_{} \overset{\text{notation}}{=} \int_C \vec{F} \cdot d\vec{r} \overset{\text{def}}{=} \int_{t=a}^{t=b} \left[\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \right] dt . \qquad (2)$$

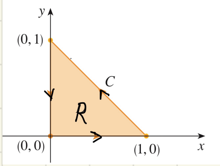
to calculate this without (Green), would need to find $\vec{r}(t)$.

Def in Green's Thm for the curve C traced out by $\vec{r}$.

1. positively oriented: if walk along C then R is to your LEFT (TL: unit circle)
2. piecewise-smooth: except for a finite number of pts, $\vec{r}$ is smooth ($\vec{r}\,'$ cont. & $\neq \vec{0}$)
3. simple: C does not intersect itself, i.e. if $a < t_1 < t_2 < b$ then $\vec{r}(t_1) \neq \vec{r}(t_2)$
4. closed: $\vec{r}(a) = \vec{r}(b)$

Recall, If $-C$ denotes "C traced out in the opposite direction, then $\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$.

Ex 1 For the triangular curve C consisting of the line segments from $(0,0)$ to $(1,0)$, followed by from $(1,0)$ to $(0,1)$, then from $(0,1)$ to $(0,0)$, evaluate the line integral $\int_C x^4\,dx + xy\,dy$.



Soln

Ex 2 For the C which is the circle $x^2+y^2=9$ traced counterclockwise, evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$.

Soln

Ex 3 Do Ex 2 but with C traced clockwise. Soln.

Using Green's Theorem to find Area

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}\,. \quad \text{(Green)}$$

If $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 1$, then

$$\boxed{\oint_C P\,dx + Q\,dy = \iint_R dA = \text{area of } R}$$

In each of the following three situations, $\underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 1}$.

(1.)	$P(x,y) = 0$	and	$Q(x,y) = x$	TL	$1 - 0$	$= 1$
(2.)	$P(x,y) = -y$	and	$Q(x,y) = 0$		$0 - (-1)$	$= 1$
(3.)	$P(x,y) = -\frac{1}{2}y$	and	$Q(x,y) = \frac{1}{2}x$		$\frac{1}{2} - (-\frac{1}{2})$	$= 1$

So (when C is parametrized with positive orientation)

$$\text{area of } R = \overset{(1)\downarrow}{\oint_C x\,dy} = \overset{(2)\downarrow}{\oint_C -y\,dx} = \overset{(3)\downarrow}{\frac{1}{2}\oint x\,dy - y\,dx}$$

↑ note also avg. of previous two

Ex 4 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Soln

$\boxed{\text{This finishes 16.4}}$