16.2 Line Integral of Vector-Valued Functions

In this section, the path $\vec{r}$ defined on $[a,b]$ parameterizes the curve C.

- In 16.1 we had: $\vec{r} : [a,b] \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^1$.

TL: $[a,b] \xrightarrow{\vec{r}} \mathbb{R}^n \xrightarrow{f} \mathbb{R}^1$.

The <u>line integral of f along C</u> is

$$\boxed{\underbrace{\int_C f\, ds}_{\text{notation}} \stackrel{\text{def}}{=} \int_{t=a}^{t=b} f(\vec{r}(t))\, \|\vec{r}\,'(t)\|\, dt}$$

- In 16.2 we have $[a,b] \xrightarrow{\vec{r}} \mathbb{R}^n \xrightarrow{\vec{F}} \mathbb{R}^n$.

The <u>line integral of $\vec{F}$ along C</u> is

$$\boxed{\underbrace{\int_C \vec{F} \cdot d\vec{r}}_{\text{notation}} \stackrel{\text{def}}{=} \int_{t=a}^{t=b} \left[\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \right] dt}$$

- So if $\vec{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \rangle$ with $F_i : \mathbb{R}^3 \to \mathbb{R}^1$ and $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, then

$$\int_C \vec{F} \cdot d\vec{r} \stackrel{\text{def}}{=} \int_{t=a}^{t=b} \left[\langle F_1, F_2, F_3 \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \right] dt$$

$$\stackrel{\text{more notation}}{=} \int_C F_1\, dx + F_2\, dy + F_3\, dz$$

where for each F_i (i.e. for F_1, F_2, and F_3):

$$\boxed{\begin{aligned} \int_C F_i\, dx &= \int_{t=a}^{t=b} F_i(\vec{r}(t))\, x'(t)\, dt \\ \int_C F_i\, dy &= \int_{t=a}^{t=b} F_i(\vec{r}(t))\, y'(t)\, dt \\ \int_C F_i\, dz &= \int_{t=a}^{t=b} F_i(\vec{r}(t))\, z'(t)\, dt \end{aligned}}$$

Similarly for $\mathbb{R}^2$ w/ $\vec{F} = \langle F_1, F_2 \rangle$ and $\vec{r}(t) = \langle x(t), y(t) \rangle$.

Recall Definition from 12.3 (dot product)

The work done by a constant force $\vec{F}$ acting along a displacement of $\vec{D} = \vec{PQ}$ is

$$\boxed{W = \vec{F} \cdot \vec{D}}$$

Ex 1 When the force is not constant, define work as a line integral.

Def. The work done by a (varying) force $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ in moving an object along a curve C, traced out by the path $\vec{r} : [a, b] \to \mathbb{R}^3$, from the point $r(a)$ to the pt. $r(b)$, is

$$W = \int_C \vec{F} \cdot d\vec{r} \overset{\text{i.e.}}{=} \int_{t=a}^{t=b} \left[\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \right] dt$$

Ex 2 Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y, z) = \langle xy, yz, zx \rangle$ and C is traced out by the path $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ for $0 \le t \le 1$.

TL: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Soln

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=a}^{t=b} \left[\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \right] dt$$

$\vec{F} = \langle x\ y,\ y\ z,\ z\ x \rangle$

$$= \int_{t=0}^{t=1} \langle (t)(t^2), (t^2)(t^3), (t^3)(t) \rangle \cdot \langle 1, 2t, 3t^2 \rangle \, dt$$

$$= \int_{t=0}^{t=1} (t^3 + 2t^6 + 3t^6) \, dt$$

$$= \int_{t=0}^{t=1} (t^3 + 5t^6) \, dt = \left. \frac{t^4}{4} + \frac{5t^7}{7} \right|_{t=0}^{t=1} = \boxed{\frac{27}{28}}$$

Ex 3 Using the $\vec{r}$ from Ex 2, evaluate $\int_C xy \, dx$.

Soln

TL: $F(x,y,z) = xy$

$$\int_C xy \, dx = \int_{t=a}^{t=b} \underbrace{F(\vec{r}(t))}\, \underbrace{x'(t)}\, dt = \int_{t=0}^{t=1} \underbrace{(t)(t^2)}\, \underbrace{(1)}\, dt = \int_{t=0}^{t=1} \underbrace{t^3}\, dt = \boxed{\frac{1}{4}}$$

Ex 4 Compare Ex 2 and Ex 3. What do you notice?

Hint: Find Ex 3's ⎵ and ⎵ and ⎵ in Ex 2.

Ex 5 Find a parameterization of the line segment from from the point $P = (p_1, p_2, p_3)$ to $Q = (q_1, q_2, q_3)$.

Soln $P \longrightarrow Q$

$$\vec{PQ} = \langle q_1 - p_1,\ q_2 - p_2,\ q_3 - q_3 \rangle \quad \Rightarrow \quad \begin{aligned} x(t) &= p_1 + t\,(q_1 - p_1) \\ y(t) &= p_2 + t\,(q_2 - p_2) \\ z(t) &= p_3 + t\,(q_3 - p_3) \end{aligned}$$

$\Rightarrow r(t) = \langle p_1 + t(q_1 - p_1),\ p_2 + t(q_2 - p_2),\ p_3 + t(q_3 - p_3) \rangle$ for $0 \le t \le 1$.

Ex 6 Evaluate $\int_C y\,dx + z\,dy + x\,dz$ where C consists of the line segment C_1 from $P = (2,0,0)$ to $Q = (3,4,5)$ followed by the vertical line segment C_2 from $Q = (3,4,5)$ to $R = (3,4,0)$.

Soln BTW: Here $\vec{F}(x,y,z) = \langle y, z, x \rangle$ and

$$\int_C y\,dx + z\,dy + x\,dz = \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} =$$

$$= \int_{C_1} y\,dx + z\,dy + x\,dz + \int_{C_2} y\,dx + z\,dy + x\,dz$$

1. For C_1: $r(t) = \langle 2 + 1t,\ 0 + 4t,\ 0 + 5t \rangle = \langle 2+t, 4t, 5t \rangle$ for $0 \le t \le 1$.

$$\int_{C_1} \underbrace{y}_{(4t)}\,\underbrace{dx}_{(1)} + \underbrace{z}_{(5t)}\,\underbrace{dy}_{(4)} + \underbrace{x}_{(2+t)}\,\underbrace{dz}_{(5)}$$

$$= \int_{t=0}^{t=1} \left[(4t)(1) + (5t)(4) + (2+t)(5) \right] dt = \int_{t=0}^{t=1} \left[29t + 10 \right] dt$$

$$= \left(\frac{29t^2}{2} + 10t \right) \Big|_{t=0}^{t=1} = \frac{29}{2} + 10 = \frac{49}{2}$$

2. For C_2: $r(t) = \langle 3 + 0t,\ 4 + 0t,\ 5 + -5t \rangle = \langle 3, 4, 5 - 5t \rangle$ for $0 \le t \le 1$.

$$\int_{C_1} \underbrace{y}_{(4)}\,\underbrace{dx}_{(0)} + \underbrace{z}_{(5-5t)}\,\underbrace{dy}_{(0)} + \underbrace{x}_{(3)}\,\underbrace{dz}_{(-5)}$$

$$= \int_{t=0}^{t=1} \left[(4)(0) + (5-5t)(0) + (3)(-5) \right] dt = \int_{t=0}^{t=1} -15\,dt = -15t \Big|_{t=0}^{t=1} = -15$$

3. $\int_C y\,dx + z\,dy + x\,dz = \frac{49}{2} - 15 = \frac{49}{2} - \frac{30}{2} = \boxed{\frac{19}{2}}$.